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DOCTORAL THESIS

Hourly Price Forward Curves for Electricity Markets

Construction, Dynamics and Stochastics

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Abstract

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Hourly Price Forward Curves for Electricity Markets

by Audun Sviland Sætherø

This thesis is about the construction of the hourly price forward curve (HPFC) for electricity prices. The HPFC is the basis for many valuation problems energy companies face, as it determines the price they can take for the delivery of electricity on an hourly level. The HPFC combines the information from historical spot prices as well as other exogenous variables and the information of the currently observed Futures products to construct a curve giving a price for electricity with delivery at some point in the future. We start the thesis with a comparison of three different methods for the construction of the HPFC, two methods from the literature and one novel method based on a joint optimization approach of both the seasonality pattern and the fitting to the observed Futures prices. This section is meant as a review section and as the starting point of our further research. Such a comparison between different methods is not currently present in the literature. By comparing the different methods we get a greater insight in the pros and cons of the different methods. These pros and cons are hard to observe while one only consider a single model, which seems to be the standard from the literature. We do not conclude which of the methods we compare is the best, as they all have their individual strengths and weaknesses. By understanding the individual models we show how we can extract the strengths from each model combining these strengths in one model.

In the second part of the thesis we study the adjustment part of the curve, or how we fit the HPFC to the observed Futures prices. We start by constructing a set of price forward curves (PFCs) for 2015 fitted to Futures prices observed in 2014, resulting in 252 individual curves. We keep the seasonality curve constant for each set of PFCs. By observing how these curves change in time, we get new insights on what are natural traits of the adjustment function. We are mainly interested in what happens when the price of a certain product is changing, and what happens when a product is cascaded into several products with shorter delivery periods. We therefore investigate the relationship between the PFC and the individual Futures products and observe this relationship is linear when the number of products remain constant. We can therefore easily investigate the effect a change in each Futures product has on the curve, as the linearity means that this effect is independent of the current price level of the observed products. We also observe that in models where the number of parameters are dependent on the number of observed Futures products, there exists a theoretical arbitrage opportunity when new products are included in the market. By investigating how the PFCs change when the Futures prices change, we get new information, which can not be observed when only considering one PFC. Such an analysis of the derivative of the PFC with respect to the Futures prices is currently not present in the literature. Benth and Paraschiv, 2017 do a similar analysis where they analyze a set of constructed HPFCs for a longer time period, but they do not consider the relationship between the Futures and the resulting HPFCs, they consider the resulting curves as a random field and make a statistical analysis of this random field. They later fit a spatio-temporal dynamical model to this data set.

In the last part of our thesis we construct a stochastic model for the PFC which is consistent with a PFC that is linear with respect to the Futures prices. Most studies on stochastic modeling of Futures products only take into account products with set delivery lengths, but this will not be in accordance to how we price our PFC. If one wants to look at the distribution of a quarterly product, one might in the future need to consider the sum of three monthly products, while today one can only trade in the quarterly product. We propose a framework where we model the Futures prices by an Ornstein-Uhlenbeck process, where the distributions of all such products are consistent to each other and to how we construct our PFC. The main contributions in this section consist of how the parameters of the different processes compare to each other, to the seasonality curve and to the adjustment function. Such a framework, where we construct a stochastic model for the different Futures products that is consistent to with respect to how we construct the PFC is to our knowledge not existing in the literature. Benth and Paraschiv, 2017 do something similar when they fit their spatio-temporal dynamical model to their set of HPFCs, but the difference is that where they fit the model directly to the data. Our model is a transformation of the model for the Futures prices that preserves the linear relationship between the PFC and the observed Futures prices, which we studied in the previous section.

Zusammenfassung

In dieser Arbeit betrachten wir die Konstruktion von Hourly Price Forward Curves (HPFC) für Strompreise. HPFC dienen als Grundlage für viele Energieunternehmen, um den Preis der Stromlieferung auf stündlicher Basis zu ermitteln. HPFC kombinieren Informationen von historischen Spot-Preisen und exogenen Variablen mit Informationen von aktuell verfügbaren Futures-Kontrakten zur Konstruktion einer Kurve, die einen Preis für Strom mit Lieferung an einem gewissen Punkt in der Zukunft liefert. Zu Beginn der Arbeit vergleichen wir drei unterschiedliche Konstruktionsmöglichkeiten. Zwei Methoden stammen aus der Fachliteratur. Eine weitere, neuartigere Methode basiert auf dem gemeinsamen Optimierungsansatz der Saisonalität als auch der Anpassung an Futures-Preisen. Ein derartiger Vergleich von unterschiedlichen Methoden ist derzeit nicht in der Literatur vorhanden. Durch den Vergleich erhalten wir einen besseren Einblick in die Vor- und Nachteile der einzelnen Methoden. Diese Vor- und Nachteile sind schwer zu erkennen, betrachtet man nur ein Modell, welches man als Standardmodell der Literatur ansieht. Dieses Kapitel soll als Überblick und Ausgangspunkt der weitern Forschung dienen. Wir erstellen kein abschließendes Ranking, da jedes Modell seine individuellen Stärken und Schwächen besitzt. Nach den Analysen der einzelnen Modelle zeigen wir vielmehr, wie man die jeweiligen Stärken extrahiert und zu einem einzigen Modell kombiniert.

Im zweiten Teil der Arbeit betrachten wir die Adjustierungsmöglichkeiten der Kurve, bzw. wie wir die HPFC an beobachtete Futures-Preise anpassen können. Wir starten mit der Konstruktion einer ganzen Reihe von Price Forward Curves (PFCs) basierend auf den Daten eines Jahres, wobei wir die Saisonalität konstant halten. Durch das Beobachten der Kurvenveränderungen mit der Zeit kommen wir zu neuen Einsichten hinsichtlich der natürlichen Merkmale der Einstellfunktion. Unser Interesse liegt darin zu erfahren, was passiert, wenn sich der Preis eines bestimmten Produktes ändert oder wenn ein Produkt in mehrere Produkte mit kürzeren Lieferperioden kaskadiert. Zu diesem Zweck untersuchen wir den Zusammenhang zwischen der PFC und dem individuellen Futures-Produkt. Wir beobachten, dass ein linearer Zusammenhang besteht, wenn die Anzahl der Produkte konstant bleibt. Wir können daher leicht die Wirkung einer Veränderung eines jeden Future-Produkts auf die Kurve untersuchen, da Linearität bedeutet, dass dieser Effekt unabhängig vom aktuellen Preisniveau der beobachteten Produkte ist. Des Weiteren können wir erkennen, dass für Modelle, bei denen die Anzahl der Parameter von der Anzahl der beobachteten Futures-Produkten abhängt, eine theoretische Arbitrage-Chance besteht, wenn neue Produkte in den Markt eingebracht werden. Durch die Untersuchung der Änderung von PFCs, falls sich die Future-Preise ändern, erhalten wir neue Informationen, die man durch isolierte Betrachtung der PFC nicht beobachten kann. Solch eine Analyse der PFC in Bezug auf Futures ist bisher nicht unternommen worden. Benth and Paraschiv, 2017 führt eine ähnliche Analyse durch, wobei Sie eine Menge an konstruierter HPFCs über einen längeren Zeitraum analysieren. Allerdings betrachten Sie nicht den Zusammenhang zwischen den Futures und den resultierenden Kurven. Stattdessen betrachten Sie die resultierenden Kurven als Random Field und führen statistische Untersuchungen an diesem durch. Später passen Sie ein räumlich-zeitliche dynamisches Modell an diesen Datensatz an.

Im letzten Teil der Arbeit konstruieren wir ein stochastisches Modell für die PFC, welches konsistent mit einer PFC ist, die linear von Futures-Preis abhängt. Die meisten Studien über stochastische Modellierung von Futures-Produkten betrachten nur Produkte mit festgelegten Lieferlängen. Dies steht allerdings nicht im Einklang mit

unserer Vorgehensweise. Wenn wir die Verteilung eines vierteljährigen Produktes betrachten wollen, ist es eventuell notwendig, die Summe der drei dazugehörigen monatlichen Produkte zu betrachten. Wir schlagen ein Modell vor, in dem wir die Futures-Preise durch einen Ornstein-Uhlenbeck-Prozess modellieren, bei der die Verteilung aller dazugehörigen Produkte konsistent ist und zu der wir die PFC konstruieren können. Der Hauptbeitrag in diesem Abschnitt bestehet darin, zu untersuchen, wie sich die Parameter der verschiedenen Prozesse zueinander, zur Saisonalität und zur Anpassungsfunktion verhalten. Ein derartiges Framework, dass aus einem stochastischen Modell für Future-Produkte besteht, welches sich im Einklang zur PFC befindet, wurde nach unserem Stand der Dinge bisher nicht in der Literatur behandelt. Benth and Paraschiv, 2017 tuen etwas ähnliches, wenn Sie in Ihrer Arbeit das räumlich-zeitlich dynamische Modell an Ihre HPFCs anpassen. Der Unterschied besteht aber darin, dass Ihr Modell direkt an Daten angepasst ist, während unser Modell eine Transformation des Modells für Futures-Preise beinhaltet, welches die lineare Beziehung ausnutzt.

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Audun Sviland Sætherø
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List of Abbreviations

(H)PFC	(**H**ourly)**P**rice **F**orward **C**urve Gives the price of a commodity (electricity), with delivery in the future, as seen today.
Futures price	Price of a Futures contract traded at an exchange covering a specific time period.
Forward price	Same as Futures price but not necessarily traded at an exchange, price is given from the HPFC.
Adjustment Function	Function modeling the difference between the seasonality curve and the PFC.
Spillover effect	Effect a change in a certain Futures price has on the PFC outside of the area this product covers. We might use the term adjustment curve as well, as the Fleten method is not modeled by a function.
Fleten model	The model for the adjustment function proposed in Fleten and Lemming, 2003.
Benth model	The model for the adjustment function proposed in Benth, Koekkebakker, and Ollmar, 2007.
Novel Model	Our novel model for the adjustment function, based on a combined least squares approach on a trigonometric spline.
Granularity	Refers to the number of Futures products observed, a finer granularity means we observe more products. Typically observing 3 monthly products instead of one quarterly.
Shot/Long end of PFC	Corresponds to the start/end of the PFC, typically first/last month for a one year curve, or last year for a five year curve.

List of Symbols

$s(t)$	seasonality curve, t=time in future.
$f(t)$	forward curve, as seen today, t=time in future.
V_j^n	Set of n Futures products observable at day j. j the we observe the products n is the number of products observed
$\hat{V}_j^n$	Set of n implied Futures products given from the PFC constructed at day j
$f_j(i, V_j^n)$	forward price for day i, as seen from day j with V_j^n as the set of Futures products observed
$d_{i,j}^k$	Derivative of the PFC for day i as seen from day j with respect to product k.
$S_j^k(T_i^s, T_i^e)$	Spillover effect from Futures product k on time period $[T_i^s, T_i^e)$. j is the day the Futures products are observed.
F	Futures product covering the whole period.
F_i	Futures product i, which covers a subset of F.
F_i^j	Futures product j in Futures product i, obtained by splitting F_i.
a_i	Sensitivity of F_i with respect to F Similar to $d_{i,j}^k$, but for longer periods.
$a^{i,j}$	Sensitivity of F_i^j with respect to F.
$a_k^{i,j}$	Sensitivity of F_i^j with respect to F_k.
c_i	Length of Futures product F. Correspondingly for c_i^j.
s_i	Seasonality curve corresponding to period i. Correspondingly for s_i^j.

Chapter 1

Introduction, background and summary

1.1 Introduction

The market for trading electricity differs from other commodity markets as electricity itself is not a commodity that is effectively storable in a large sense. Therefore, the trading of electricity and the build up of financial markets concerning electricity differs from what we observe for other financial markets. This non-storability of electricity means that the only way to hedge against price uncertainty in the future is by buying Futures contracts promising the delivery of electricity, instead of buying and storing the commodity as is an option in other markets. Contracts for delivery of electricity in the future often have low granularity, and if one wants to buy electricity more than a year in advance, one can often only trade contracts covering the whole year. The specifics of what contracts are traded at the different times, differ from market to market, we will focus on the German market in this thesis, but the proposed ideas are applicable for markets in other countries as well.

Because the Futures contracts only offer delivery of electricity over longer time periods, there exists a need for trading electricity for smaller time periods. Because of this we need an over the counter (OTC) market, where one can trade contracts offering the delivery of electricity for periods down to one hour, several years in advance. These contracts need to be priced, and the pricing of these are done with what is called the Hourly Price Forward Curve (HPFC). The HPFC is basically the price of electricity with delivery in the future as seen today, and is computed internally in the different electricity companies. As the methods for computing the HPFC differs from the different firms, the curves will differ and the firms will want to keep information about how they construct their curves secret. Therefore, we will focus on the HPFC at a theoretical level, as we can't compare with what is done for companies constructing the curve for actual trading. The curves of the different might differ on an hourly level, but all curves will need to average out to the same price over the time period where a Futures price is traded to avoid arbitrage opportunities.

This thesis will in general focus on the construction of the Price Forward Curve (PFC)[1], and everything concerning the PFC. The PFC consists of two parts, the first is a seasonality curve which represents how one typically expects the prices to distribute throughout the year. The second part is the adjustment function, which

[1] We will mostly discuss the curve for a daily granularity, and therefore use the term PFC instead of HPFC

makes sure our PFC is consistently priced with regard to the observed Futures products, to avoid arbitrage possibilities. To a large extent, the thesis will be about the adjustment function, and not the seasonality curve.

1.2 Electricity Markets

In this thesis we will use market data from the German electricity market, and therefore the market specifications follows those of the German market, that being said, other electricity markets are to a large extent organized in a similar if not completely equal way. In this section we will describe the German electricity market, and the contracts traded here, the information used comes from (ref: https://www.eex.com/en/trading/rules-and-regulations/regulated-market). The German market is covered by the EEX (European Energy Exchange), and here one can trade in energy and electricity for the markets in France, Germany, Austria and Switzerland
The trade of electricity is usually divided into three different parts, the intraday-market, the day-ahead market and the Futures market.

1.2.1 Intraday market

According to the EPEX spot website, intraday trading is defined as:

"Electricity traded for a delivery on the same or on the following day on single hours, 15-minute periods or on block of hours. Each hour, 15-minute periods or block of hours can be traded until 30 minutes before delivery begins. Starting at 3pm on the current day, all hours of the following day can be traded. Starting at 4pm on the current day, all 15-minute periods of the following day can be traded."

There are also standardized blocks that can be traded, which are the Baseload hours covering hours 1 to 24, and the Peakload covering hours 9 to 20 on every week day (Mon-Fri), but users can also give bids for specialized blocks after demand. Trading on the intraday market is done 24 hours a day, 7 days a week.

During the last years, the trading activity in the intraday market has gone up as a result of increased renewable energy, which leads to uncertain production of electricity. In 2016 the intraday market totalled out at 61 TWh, compared to only 11 in 2010, but compared to the day-ahead volume of 467TWh it is still small. This and the fact that the intraday market is still mostly used for correcting incorrect production forecasts of renewable energy, making it not that relevant for the calibration of the PFC. Therefore, we will not use intraday data in this thesis, and when we talk about spot prices we will mean the day-ahead prices.

1.2.2 Day-Ahead market

In the Day-Ahead market electricity is traded for the next day, and is set up in the form of an auction linking bid and ask prices, either for single hours, or block contracts specifying a set of hours. The auction end at 12.00 pm the day before delivery, and takes place every single day throughout the year.

Members of the EPEX SPOT market, can if they have a Futures product with financial settlement, they can send a bid for the day ahead auction asking for a physical fulfilment of the option instead. In Germany, where Futures products are only traded with financial fulfilment one can in this way get physical delivery of the Futures product instead of financial fulfilment.

In our thesis we will use the day ahead prices from 2011-2013 to calibrate our seasonality curve.

1.2.3 Futures Market

In the Futures market one trade options with financial fulfilment for electricity, meaning one get the difference between the average price for electricity on the spot market for the relevant period and the price of the Futures product. As mentioned earlier, one can opt for physical delivery, if one is a member of the EPEX SPOT market.

For the German market the maximum number of products one can trade in, is as follows:
Day Futures: The respective next 34 days.
Weekend Futures: The respective next 5 weekends.
Week Futures: The current and the next 4 weeks.
Month Futures: The current and the next 9 months.
Quarter Futures: The respective next 11 full quarters.
Season − Futures: The respective next 6 full seasons (Season Future).
Year Futures: The respective next 6 full years (Year Future).

The exact number of tradeable maturities is determined by the Management Board of the Exchange and announced before implementation. When we construct our curve, we will only use a subset of these products, as not all products, even if they are traded, will be liquidly traded, meaning certain products are only traded a couple times during a month. We will therefore only use the products closest to delivery. It is worth noting that some of the Futures products coincide, so at one point one can chose to trade in either the three first months, or the first quarter, for example.

The price of the Futures Products, as for the spot prices, are denoted in Euro/MWh, and the quoted price is the average price for the relevant period. We will in the following use the term Futures price as the price of a Futures product, where we will specify the period when needed. We will say M1 or January Futures price for the price of the Futures product covering January, or Q1 Futures price for the price of the product covering January to March.

As we sometime observe the first monthly Futures product and the quarter product covering this month as well, we will split these products into one month and a two-month product, and we will call the products M1 and M23 if one is speaking about the first quarter, and equivalently for the other quarters. The main focus of this thesis is to study the (Hourly) Price Forward Curve (H)PFC for electricity prices. We will in general not look at the curve at an hourly granularity, and we will therefore for the most part use the notation PFC. Some of what we discuss will be applicable for forward curves for other commodities, but it will be aimed at electricity markets.

The data used will come from the German electricity market, but the main point of the thesis will be a general discussion about how to construct this curve, and therefore the hope is that the results here can be applicable for other markets as well.

1.3 Contribution and Structure of the Thesis

The main point of this thesis is not to present some sort of blueprint on what is the best way to construct the price forward curve (PFC), but rather give some understanding to the wast amount of literature currently available on the subject. The construction of the HPFC is usually split into three different parts

- construct a seasonality curve representing typical characteristics of electricity prices
- adjusting the seasonality curve, making it arbitrage free to the observed Futures prices
- apply the hourly profiles to the stochastic model in an arbitrage-free way.

In certain studies they do a combined construction of daily and hourly profiles, and thereafter making the curve arbitrage-free. Many studies typically take the seasonality patters from some known method from the literature and proposes a novel method for the adjustment function. Or they take all parts of the PFC from the literature, and do some statistical survey of the resulting curves. The problem with this approach, is that it is hard to pinpoint where the different weaknesses of the resulting curve comes from. We want to study each part in detail, and by this understand the weaknesses of the different parts. As the PFC will change as we come closer to delivery, we also want to study how the PFC should change in time.

By comparing different models, and investigating each part individually we want to understand the whole curve, and how it works, giving us a better idea of how to construct it. By seeing why certain undesirable features are present in certain model, but not in others, we get insight in how the models can be changed to remove these features, instead of rejecting the whole model.

1.3.1 Structure of the Thesis

The thesis will consist of three parts: In section 2 we will explain the construction of the HPFC, where we will focus on two methods from the literature as well as a novel method. In section 3 we will investigate the dynamics of the HPFC, seeing how it changes in time when we get closer to delivery and the Futures prices change. In section 4 we will use the results we have gained to make a framework for a stochastic model for the HPFC.

Each part of the thesis is structured in the same manner. We start with an introduction and motivation of the main problem of the section, we also give an overview of what is previously done in the literature. Thereafter, we follow up with our research, and how that differs from what is done in the literature. At last we give an conclusion of our research and possible extensions of our work for future research.

Construction of (H)PFC

The first part of the thesis is a review of how to construct the HPFC, where we discuss several different methods. Such a review comparing different methods is currently not in the literature, as most studies focus on the making new methods for the construction of the HPFC. This section mostly discusses what features the HPFC should have, and which of the proposed model have these features. We also test our curves against data, both an in-sample and out of sample test, but the tests do not give a concluding answer to which model is the best.

This section serves as the background for the two remaining parts of the thesis, but does not itself contain a lot of results, as it is as said mostly a review section. In this section we talk about both hourly price forward curves (HPFCs) and price forward curves (PFCs). In the next two sections we will focus on how the Futures prices affect our curves, and as we do not work with peak and off-peak products, our hourly profile will remain unchanged. Therefore, we will focus on daily prices and we will mostly use the term PFC instead of HPFC.

Dynamics of the PFC

In this section we analyze how the PFCs change when Futures prices used as input change. We first construct a set of 252 PFCs for each method discussed, and observe how they evolve in time. By observing this for all three methods we want to understand what characteristics are natural and which are not.

We afterwards study the relationship between our Futures prices and the corresponding PFC and show that this relationship is linear when the number of observed Futures products is constant. As this is linear, we can find a derivative of our PFC with respect to the Futures prices saying how the price of any time period changes when the different Futures prices change. We study this derivative, and try to give reasoning which characteristics are fitting and which should be rejected. We also give suggestions on how a new adjustment function could be constructed based on our findings.

In this thesis we work with a PFC covering only one year. By understanding the effect the Futures prices have on the curve, we can also say how such a curve will differ from a curve covering 2 or more years in the different methods. From this we will give some considerations about how the curve should react to the inclusion of more products in the long end of the curve.

Stochastic model for PFC

In the final part of our the thesis we develop a framework for a stochastic PFC using the linear relationship between our Futures prices and the PFC showed earlier. We start with a considering the stochastic PFC as a linear combination of the observed Futures products, where the Futures products are modeled by some stochastic differential equation. In the literature there are several studies on how to model Futures prices with SDEs, but most of these studies work with models where the number of observed products remain constant.

In this thesis we consider how such a framework should work when the Futures products are cascading, as we now might only observe a quarterly product, but in the future might observe the individual monthly products. This means we need to three independent products in the future, but only one now. We therefore propose using processes whose distribution is infinitely divisible, where we mostly work with a classic Ornstein-Uhlenbeck process driven by a Brownian motion.

The results of this section of the thesis consist of how the parameters of the processes for the individual products will compare to each other, and how they compare to the seasonality curve, and adjustment function from our PFC. This gives us a framework for computing a probability distribution for the PFC in the Future, even when we in the future will observe more Futures products than we currently do. This probability distribution will be consistent with how we fit our PFC to the observed Futures prices for all models that have a linear relationship between the PFC and the Futures prices.

Chapter 2

Construction of the HPFC

2.1 Introduction

As most industrial costumers of a utility are heavily dependent on electricity for production purposes and have very little flexibility in demand, they need to minimize the risk induces by highly-volatile electricity prices. Similarly, a producer of electricity will be interested to hedge this risk and thus to secure the level of the price today for the delivery of electricity at a future period of time. This becomes highly relevant, since electricity suppliers must cover their production costs and, in addition, electricity is non-storable and it must be consumed immediately as it is produced. Consumers and producers of electricity will thus ensure the (continuous) delivery of electricity over a certain period of time in the future. Futures contracts for electricity are however standardized for delivery of power over a limited set of delivery periods: Over one week, one month, one quarter or one year. There is a limited number of traded Futures contracts at EPEX (The European Power Exchange): weekly, monthly, quarterly and yearly, which restricts the flexibility of market participants to adjust to price levels which typically differ for different hours of one day, weekdays and seasons. For this purpose market participants use the information from PFCs to read the fair price for individual hours. This becomes highly relevant for example for electricity consumers with specific load profiles, where the consume of electricity is concentrated at specific hours.

Updated HPFCs are of particular interest nowadays especially in countries like Germany, where there has been a continuous increase of the in-feed of wind and photovolatic for the electricity production (Erni, 2012 and Hildmann, Ulbig, and Andersson, 2013). Renewable energies are highly volatile and difficult to forecast accurately. Thus, weather updates are observed until short before the delivery period and weather forecasting errors are incorporated in the price formation process in the intraday electricity market Kiesel and Paraschiv, 2017, which implies a high uncertainty around the spot price level. It is therefore relevant to have access to accurate expectations of prices for each hour of the day, which is the goal of hourly price forward curves. The standardization of forward prices along the price forward curve is a hedge against the volatile spot electricity prices and allows market participants to plan better their production and balance out consumption in the future.

While on the forward market the electricity is traded for future delivery periods, the day-ahead and intraday markets allow for the possibility to correct the long-term production schedule of power plants (Delta Hedging) and to adjust for the residual load profiles on an hourly or quarter-hourly basis Kiesel and Paraschiv, 2017.

For the construction of PFCs we typically incorporate the information about market expectation from the observed futures prices and the deterministic seasonal effects of electricity prices. There are several methods in the literature for the construction of the PFC Fleten and Lemming, 2003, Benth, Koekkebakker, and Ollmar, 2007, Paraschiv, Fleten, and Schürle, 2015 and Caldana, Fusai, and Roncoroni, 2017. Which differ among each other with respect to the method chosen for the seasonality shape, to the smoothing component, and with respect to the methodology of getting arbitrage free curves. The typical seasonality patterns of electricity prices contain yearly, weekly and daily patterns which determine ultimately the shape of the demand profile for electricity. In this study, we discuss the different mathematical models used for the construction of the seasonality shape. And we discuss the effect of one or another method for the derivation of the seasonality shape on the final resulting PFC. We implemented the existing methods of Fleten and Lemming, 2003, Benth, Koekkebakker, and Ollmar, 2007, Paraschiv, Fleten, and Schürle, 2015 and discuss comparatively the features of the generated PFCs. On top of this we propose a novel method for the construction of the PFC. The main feature of our model is that we do not treat the seasonality shape exogenously, as it is done in Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007, but we formulate a more flexible optimization model, where we simultaneously shape and align the curve to the level of the observed Futures prices in a joint optimization procedure. This is insofar important, since it allows a more direct comparison of PFCs in different energy markets with slightly different patterns of the seasonality curves. We will test and compare the selective models with respect to their ability to replicate and forecast the observed electricity prices, which is an additional contribution of this study to the existing literature on PFCs.

The rest of the section is organized as follows: In Section 2.2 we give a review of the different approaches used for the construction of the PFCs. In Section 2.3 we make a comparative assessment of these modeling approaches. In Section 2.4 we compare the different estimated curves with respect to the observed spot prices and Section 2.5 concludes.

2.2 A Review of Modeling Approaches for Price Forward Curves

All methods to construct the HPFC follow in large part the same generic principles. We will compare the different methods used, and explain the strengths and weaknesses of the proposed approaches. We compare the methods from Paraschiv, Fleten, and Schürle, 2015, Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 together with a novel model based on trigonometric splines.

The construction of an HPFC is usually split into three parts.

- First we construct the seasonal curve, which indicates how the prices are distributed throughout the year.
- The second step is to adjust this curve by making it arbitrage free with respect to the observed Futures. We will call this the adjustment part of the HPFC.
- As a third step, to get hourly prices, we will need to apply an hourly profile to the daily prices.

The seasonality curve is constructed by fitting appropriate periodic functions to historical spot prices. We assume that the typical seasonality patterns are recurrent each year. The seasonality curve will contain yearly, weekly and daily components. The seasonal patterns occur due to weather conditions or economic and business activities.

Yearly Seasonality: This is related to natural phenomenons, as different temperatures between summer and winter seasons, which determines a different demand pattern for electricity. The yearly seasonality is also related to vacation and holiday periods where economic activity and thus the use of energy is reduced.

Weekly Seasonality: Electricity prices are generally higher during the week, when the economic activity is intense, then during the weekend. Therefore, one typically observes a jump in prices when going from working day to weekend/holiday, therefore we will include dummy variables for the different days of the week, to correct for this pattern.

Daily Seasonality: The daily seasonality of electricity prices is determined by the economic activity within one day. Typically one observes lower prices during the night, prices start increasing during the morning hours and reach a peak around noon. It has been empirically observed that the noon peak has flattened over time because of the increasing in-feed of renewable energies Paraschiv, Bunn, and Westgaard, 2016. In winter one typically observes a second evening peak in the German market, related to the extra need of heating as people come home from work.

The typical yearly cycles are typically modeled by trigonometric functions which produces a smooth shape. The other patterns of the seasonality shape related to economic activity (weekly and daily) are typically modeled by dummy variables. In this study we will consider three types of seasonality functions, dummy variables, Fourier series or splines.

2.2.1 Review of different functions used for the seasonality shape

We give a review of the main functions used for modeling the seasonality patterns. We refer here to both dummy variable related models and trigonometric functions.

Dummy Variables: Paraschiv, Fleten, and Schürle, 2015 model the combined yearly and weekly seasonality curve by a mixture of dummy variables and continuous variables for the cooling/heating degree days (CDD/HDD) for three different German cities, defined as follows:

$$f2_{y_d} = a_0 + \sum_{i=1}^{6} b_i D_{di} + \sum_{i=1}^{12} c_i M_{di} + \sum_{i=1}^{3} d_i CDD_{di} + \sum_{i=1}^{3} e_i HDD_{di} \tag{2.1}$$

Where a_0 can be interpreted as the mean level of the year. The rest of the terms shape the weekly (D_{di}), and the yearly cycle is modeled by dummy variables for each month (M_{di})[1], and it is further stylized by the CDD/HDD.

[1] August ist split into two, to account for holiday periods

We empirically observed that the problem with modeling the seasonality curve by dummy variables is that they mainly account for the change in the price level between months, while in reality one expects the price changes to occur more smoothly. Fleten and Lemming, 2003 cope with this problem by smoothing the HPFC by the adjustment function.

Fourier Series: Truncated Fourier series are sums of trigonometric functions of the form:

$$F_n(t) = a_0 + \sum_{i=1}^{n} [a_i \sin(i \cdot \pi \cdot t) + b_i \cos(i \cdot \pi \cdot t)]$$

and are commonly used to model cycles. The reason for this is that they have a natural periodicity, depending on their frequency. The advantage of these functions compared to dummy variables is that they are continuous, meaning there are no sudden jumps between periods. As the intergral of the trigonometric terms in the Fourier series $F_n(t)$ is equal to 0 over the period $0 \leq t \leq 2$, the constant a_0 corresponds to the mean of the year.

The number n decides how many terms to include, a higher n gives better fit to the data, but also increases the chance of overfitting. The use of Fourier series for seasonality functions is common, given their simplicity, and they are also used for other commodities. However, the pattern produced by trigonometric functions is too regular, and we can not model effectively all characteristic price changes.

As a note, it is common when fitting Fourier series to data to use only functions of the form:

$$f_1(t) = a_1 \sin(2\pi t) + b_1 \cos(2\pi t)$$

and not of the form

$$f_2(t) = a_2 \sin(2\pi t + \theta) + b_2 \cos(2\pi t + \theta)$$

since the trigonometric identities

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

make these functions equivalent. Since the first form is linear in the parameters, this function can be fitted using ordinary least squares.

In Figure 2.1 we show an example of an Hourly profile estimated by Fourier series. The fit to the data seems in general good, apart from hour 7 and hour 23. At hour 7 the mean price is much lower than estimated by the Fourier series, which is probably an effect of the fact that at this hour power plants are turned on to cover the typical increase in the demand during the morning, resulting in an overproduction at that hour, driving the prices down. In a similar way one observes that the prices increase at hour 23, which can be interpreted by the fact that power plants shut down.

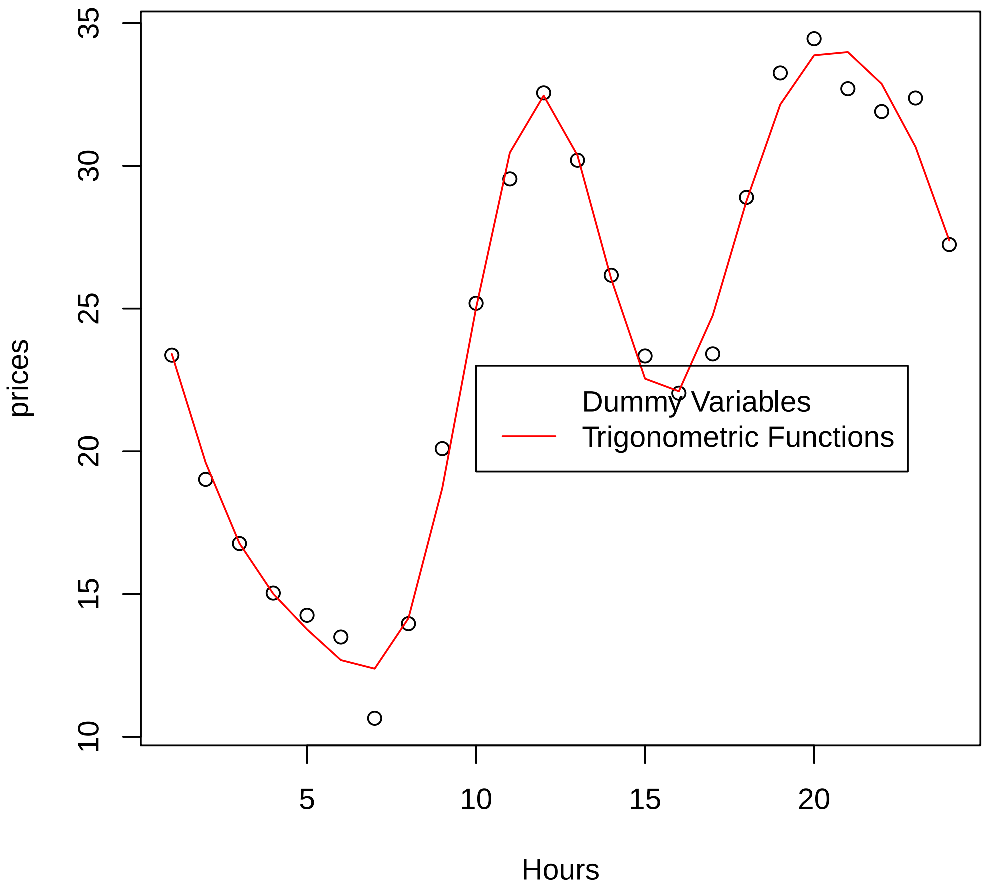

FIGURE 2.1: The circles reflect the observed mean price for each hour of the day during the years 2000-2007 and the continuous line corresponds to the fitted Fourier series of the form $F_4(t) = a_0 + \sum_{i=1}^{4} a_i \sin(2\pi it/24) + b_i \cos(2\pi it/24)$ to the same data

As the goal of a HPFC is of course not to validate already known data, but to estimate the future prices. In the following we test how daily profiles constructed by Fourier series and by dummy variables fit to observed prices during the years 2008-2015. In Table 2.1 we show the absolute and square difference of the results with the two approaches.

The test is done as follows: We obtain our estimated prices by taking the real price for each day d multiplied by the hourly profiles to get an estimated price for each hour h in day d. Then we take mean of the absolute difference between this estimate and the observed price for hour h at day d for all days in 2008-2015. The same is done for the squared differences.

$$\text{Absolute\_Error\_Hour}_h = \frac{1}{n}\sum_{d=1}^{n} |DayPrice_d \cdot HourlyProfile_h - HourPrice_{d,h}| \quad (2.2)$$

$$\text{Squared\_Error\_Hour}_h = \frac{1}{n}\sum_{d=1}^{n} (DayPrice_d \cdot HourlyProfile_h - HourPrice_{d,h})^2 \quad (2.3)$$

As one can see in table 2.1, results are inconclusive, so choosing one method over the other might not matter much for the overall fit. As observed in 2.1 there are deviations with the approximation of the Fourier series from the observed mean prices for some specific hours (hour 7 and hour 23 are examples). In such cases a combined approach with approximation with Fourier and the inclusion of specific dummy variables for hours when deviations occur will be a better approach.

Spline Functions: The idea behind spline functions is to model the different segments of the curve independently, and then put them together at the knot points to ensure suitable continuity of the curve. The advantage of the spline over dummy variables is that it generates continuous curves while it still offers greater flexibility than standard trigonometric functions. The standard approach is to use a polynomial of a certain degree, but the idea can also be transferred to trigonometric and exponential functions. This type of curve is not commonly used in the literature, but we will later introduce a novel approach using trigonometric splines for the seasonality function.

Overall one should not use one specific de-seasonalisation approach in isolation, but rather use a combination of the different techniques to reflect observed seasonal patterns in electricity prices.

Finally we should point out that it is common to use other fundamental variables in the construction of the seasonality curve. Standard variables are heating/cooling degree days, demand forecasts modeling the expected demand for electricity and fuel prices are also commonly used variables. Given the increasing in-feed from renewable energies in the system it is natural to include weather forecasts for wind and photovoltaics as additional explanatory variables. In Erni, 2012 the fundamentals of hourly spot electricity prices are derived and the same fundamentals might be reasonable drivers for the HPFCs, this is a subject of future research.

	L1		**L 2**	
	Fourier	**Dummy**	**Fourier**	**Dummy**
Hour 1	**10.46224**	10.47887	**172.57410**	173.00676
Hour 2	**10.86177**	11.10551	**220.81164**	226.57419
Hour 3	**13.52108**	13.52546	**252.97063**	253.10604
Hour 4	10.70268	**10.69568**	215.86698	**215.64835**
Hour 5	10.42771	**10.25740**	187.07475	**182.18272**
Hour 6	10.68178	**10.40417**	186.40433	**177.21438**
Hour 7	**10.75718**	11.37051	**247.72809**	263.28876
Hour 8	**10.23698**	10.30623	**174.67778**	176.73565
Hour 9	10.144247	**9.620673**	152.941690	**138.612149**
Hour 10	9.767245	**9.713098**	138.157351	**136.937834**
Hour 11	**9.919567**	10.061363	**145.067145**	146.545121
Hour 12	10.90129	**10.88703**	170.46746	**170.28211**
Hour 13	**11.48788**	11.51683	**194.35153**	194.93814
Hour 14	11.46580	**11.44212**	233.74733	**233.48758**
Hour 15	11.79799	**11.61616**	286.25036	**284.31261**
Hour 16	**11.21104**	11.22297	**253.73688**	253.91762
Hour 17	**10.35865**	10.55824	**180.02174**	184.50132
Hour 18	11.27468	**11.24566**	190.57921	**189.55610**
Hour 19	12.05114	**11.65319**	231.49264	**215.45815**
Hour 20	12.02813	**11.76629**	237.87038	**227.60371**
Hour 21	**10.32580**	10.99499	**175.48254**	195.58561
Hour 22	**9.176717**	9.689951	**139.533732**	151.995862
Hour 23	11.49978	**10.41221**	202.69297	**173.77011**
Hour 24	**9.990992**	10.057976	**163.474317**	165.134128
Mean	10.87718	**10.85844**	198.08231	**197.09979**

TABLE 2.1: Comparison of the error estimate between a daily profile estimated by truncated Fourier series and dummy variables, column one and three are the errors for the Fourier series, while two and four for Dummy variables, L1 is absolute error, while L2 is square error.

2.3 Comparative Assessment of Modeling Approaches for (H)PFCs

Once we identified the typical seasonality pattern of electricity prices, one seasonality shape can be constructed. Independently of the type of methodology used for the derivation of the seasonality shape, the historically estimated model-parameters can be used to forecast the seasonality curve, see Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007. In case of the use of fundamental variables (Erni, 2012), like weather data, renewable energies and load forecasts, forecasting models for those variables are defined. However, in the case of historically derived seasonality shapes, their forecast does not incorporate yet the market expectation. We therefore must further align the generated seasonality shape to the level of the observed Futures prices in the market, to avoid arbitrage opportunities.

2.3.1 Review of Existing Models

In the current study, we discuss two different popular approaches for the derivation of the price forward curves, namely Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 as well as a novel approach based on constrained least squares of trigonometric splines. In the two first studies the seasonality shapes have been historically derived and represent an exogenous input for the derivation of the price forward curves, while in the novel approach we suggest a combined calibration of both the seasonality and adjustment function. The two first optimization procedures have as a main objective the minimization of the distance between the seasonality curve and the resulting price forward curve under certain constraints. The curve should be arbitrage free and the constraints ensure that the average of the forward prices on the different segments on one curve meet the corresponding level of the observed Futures prices. In the two first construction methods it is assumed that the PFC $f(t)$ can be decomposed into a seasonal component $s(t)$ and a residual term $\epsilon(t)$ modeling the difference between seasonality curve and the PFC. In the novel approach we only have one term that takes into account both the seasonal and the residual term.

In the sequel we will show the mathematical formulation of the three approaches, and describe their advantages and their drawbacks. There are two ways of fitting the curve to the futures prices, either when the market is still open and we observe bid and ask spreads (ex-ante), which is done in Fleten and Lemming, 2003, or after the market closes and final prices are observed (ex-post), as done in Benth, Koekkebakker, and Ollmar, 2007. Both methods can be adjusted to either of the two approaches, and later when we compare the curves we will use the ex-post approach in both cases to make sure the comparison is done on equal terms.

In both approaches we let $\Phi = \{(T_1^s, T_1^e), (T_2^s, T_2^e), ..., (T_m^s, T_m^e)\}$ be a list of start and end dates for m average-based forward contracts. We collect all starting and end dates in chronological order (overlapping contracts are split in sub-periods). The constructed hourly price forward curve f_t replicates the currently observed market prices $F(T^s, T^e)$ perfectly, where T^s and T^e are the start and end dates for different settlement periods.

Fleten & Lemming approach: Fleten and Lemming, 2003 model the hourly price curve by combining the information contained in the observed bid and ask prices with the information about the shape of the seasonal variation.

Let f_t be the price of the forward contract with delivery at time t, where time is measured in hours, and let $F(T_1, T_2)$ be the price of forward contract with delivery in the interval $[T_1, T_2]$. They work with only bid/ask prices, which gives the constraint:

$$F(T_1, T_2)_{bid} \leq \frac{1}{\sum_{t=T_1}^{T_2} \exp(-rt/a)} \sum_{t=T_1}^{T_2} \exp(-rt/a) f_t \leq F(T_1, T_2)_{ask} \tag{2.4}$$

where r is the continuously compounded rate for discounting per annum and a is the number of hours per year. We assume later that $r = 0$, and we work with one closing price instead of the bid/ask spread.

$$\min_{f_t} \left[\sum_{t=1}^{T} (f_t - s_t)^2 + \lambda \sum_{t=2}^{T-1} (f_{t-1} - 2f_t + f_{t+1})^2 \right] \tag{2.5}$$

The parameter lambda controls for the smoothness of the curves: $\lambda = 0$ means no smoothing, and if $\lambda \to \infty$ the originally forecasted seasonality shape will be obsolete, meaning that if one constructs a PFC from two different seasonality shapes, the resulting curves will converge to the same when $\lambda \to \infty$. In Figure 2.2 we have constructed four PFCs, one with a real seasonality curve and one where the data is drawn from a normally distributed random variable. One can observe that when λ is small the resulting PFCs differ a lot, while when λ is big they are quite similar in both cases.

In Figure 2.3 we show the difference between the PFC and the seasonality shape, where we applied the Fleten and Lemming approach. In the original model of Fleten and Lemming, 2003, applied for daily steps, a smoothing factor prevents large jumps in the forward curve. However, in the case of an hourly resolution of the curves (HPFCs), Blöchlinger, 2008 (p. 154), concludes that the higher the relative weight of the smoothing term, the more the hourly structure disappears, see Figure 2.4.

Benth et al. method : In the method suggested by Benth, Koekkebakker, and Ollmar, 2007 the constructed hourly price forward curve $f(t)$ replicates the currently observed market prices $F(T_s, T_e)$ perfectly, where T_s and T_e are the start and end dates for different settlement periods:

$$F(T^s, T^e) = \frac{1}{T^e - T^s} \int_{T^s}^{T^e} f(t)\, dt, \tag{2.6}$$

where $f(t)$ consists of a seasonality curve $s(t)$ and a correction term $\varepsilon(t)$. The correction term $\varepsilon(t)$ is modeled by a polynomial spline of the form:

$$\varepsilon_t = \begin{cases} a_1 t^4 + b_1 t^3 + c_1 t^2 + d_1 t + e_1 & t \in [t_0, t_1) \\ a_2 t^4 + b_2 t^3 + c_2 t^2 + d_2 t + e_2 & t \in [t_1, t_2) \\ \vdots & \\ a_n t^4 + b_n t^3 + c_n t^2 + d_n t + e_n & t \in [t_{n-1}, t_n] \end{cases} \tag{2.7}$$

$$x^T = [a_1\ b_1\ c_1\ d_1\ e_1 \quad a_2\ b_2\ c_2\ d_2\ e_2 \ldots a_n\ b_n\ c_n\ d_n\ e_n] \tag{2.8}$$

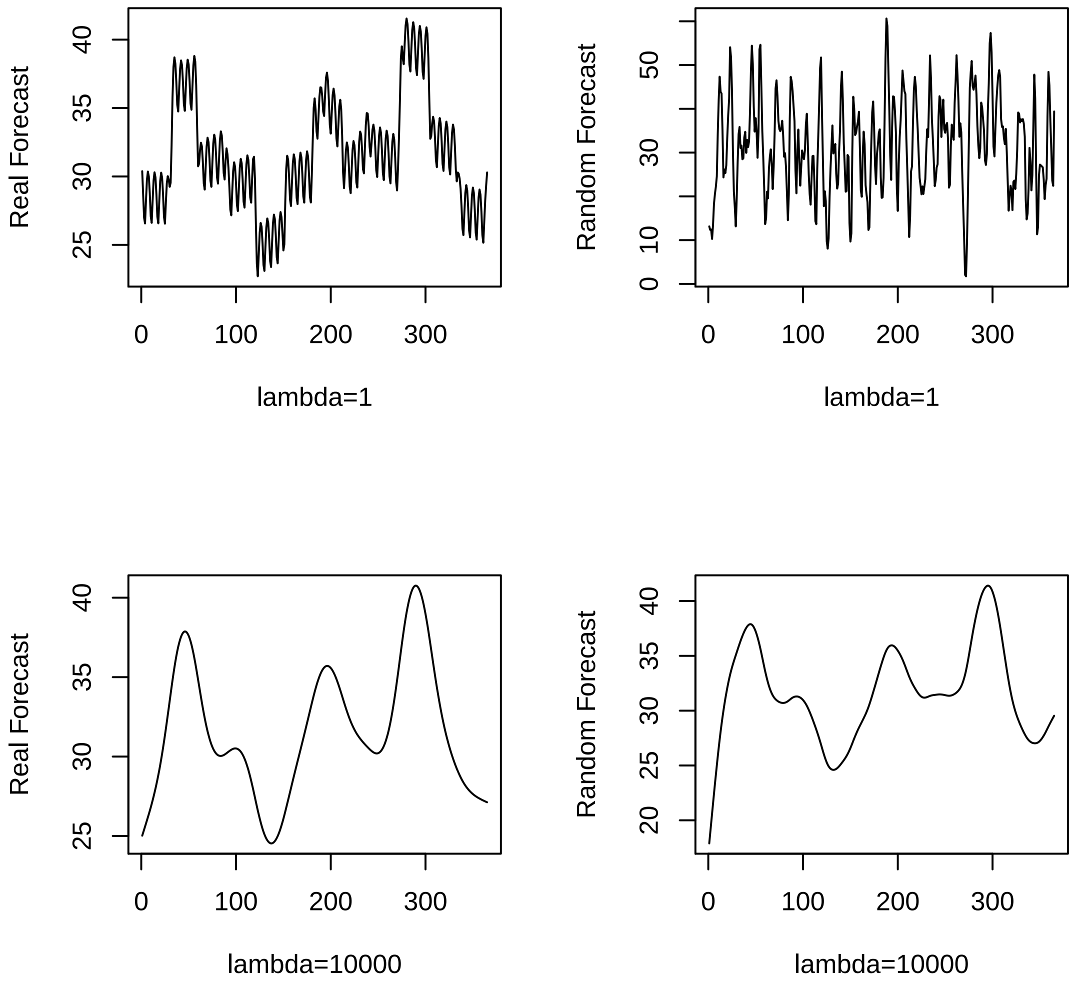

FIGURE 2.2: Showing four different PFCs constructed by the method described in Fleten and Lemming, 2003, two curves made with a forecast from dummy variables and two curves where the forecast is just drawn from a normal random variable, with $\lambda = 1$ and $\lambda = 10000$.

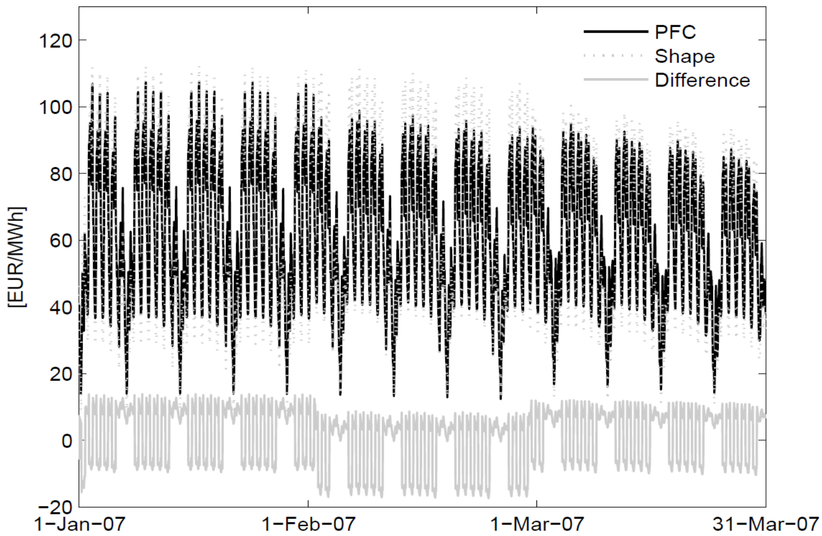

FIGURE 2.3: *Shape–HPFC* (source: Blöchlinger, 2008

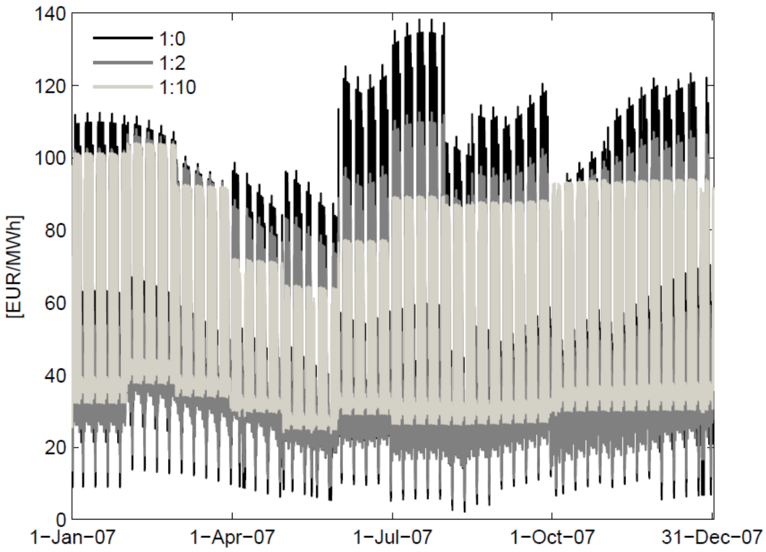

FIGURE 2.4: *Shape–HPFC* Blöchlinger, 2008

The minimization criterion that will ensure our curve has maximum smoothness is given by:

$$\min_{x} \int_{t_0}^{t_n} [\varepsilon''(t;x)]^2 dt \tag{2.9}$$

To ensure continuity and continuous derivatives throughout the periods, the following equations $(9) - (11)$ need to hold:

$$\begin{aligned}(a_{j+1} - a_j)t_j^4 + (b_{j+1} - b_j)t_j^3 + (c_{j+1} - c_j)t_j^2 \\ +(d_{j+1} - d_j)t_j + e_{j+1} - e_j =& 0\end{aligned} \tag{2.10}$$

$$\begin{aligned}4(a_{j+1} - a_j)t_j^3 + 3(b_{j+1} - b_j)t_j^2 \\ +2(c_{j+1} - c_j)t_j + d_{j+1} - d_j = 0\end{aligned} \tag{2.11}$$

$$12(a_{j+1} - a_j)t_j^2 + 6(b_{j+1} - b_j)t_j + 2(c_{j+1} - c_j) = 0 \tag{2.12}$$

To ensure that the curve is flat in the long end, we set the first derivative in the end point equal to 0, ensured by Equations (2.9)-(2.12). To account for settlement of the contracts throughout the period, one can include a function $w(r;t)$ as shown in Equation (2.14). For settlement only at the end points, one sets $w(r;t) = 1/(T_i^e - T_i^s)$.

$$\varepsilon'(t_n; x) = 0 \tag{2.13}$$

$$\tag{2.14}$$

$$F_i^C = \int_{T_i^s}^{T_i^e} w(r;t)(\varepsilon(t) + s(t))dt \tag{2.15}$$

2.3.2 Critical View

One of the differences in the two approaches is that the smoothing of the curve is done in different ways. As we have seen, in the Fleten and Lemming, 2003 approach the smoothing is done directly on the curve, while in Benth, Koekkebakker, and Ollmar, 2007 the smoothing is done on the correction term by splines.

Fleten Approach: The problem with the first approach is twofold: Firstly, the λ-parameter of the smoothing factor has an aleatory nature, there is no common sense in the literature about its size. Secondly, since the smoothing is done directly on the curve, it suppresses the daily and weekly patterns of the seasonality shape (see Blöchlinger, 2008). This can be a serious drawback when one is interested in PFCs of higher resolution.

A solution to the fact that the smoothing suppresses the weekly and hourly pattern can be to reapply these patterns after the smoothing is done. In this way we can first ensure a smooth curve, and afterwards ensure that we have a sufficient daily/weekly seasonality. We will later show evidence that such an approach gives better results when looking at the weekly seasonality.

As for the aleatory nature of the parameter λ, one solution could be to choose the smallest λ that results in a smooth enough curve. Example of what smooth enough might mean is that the largest price difference between two consecutive days are less than a pre-set number, or not too high compared to the difference between the other days.

Benth Approach: In the approach by Benth, Koekkebakker, and Ollmar, 2007 one does not suppress the weekly and daily seasonality, as the seasonality curve is not affected by the smoothing. As a result, if one uses a non-smooth seasonality curve, as is the case with dummy variables, the result will be a non-smooth HPFC. Therefore, if one is not satisfied with the smoothness observed in the seasonality curve, this approach might not be suitable.

In Figure 2.6 panel 2 we show one week on the curve generated at 01/01/2012. We observe that the typical daily seasonality of the German electricity prices is conserved. In this approach Futures prices are replicated by polynomial splines for the corresponding error term. By the construction of the model the smoothing is done continuously over the whole curve. Additional constraints ensure the continuity of the curve at the knot points, where new products become available. However, since for electricity only a limited number of Futures products are available in the market, if one is interested in the long end of the curve, where only yearly products are available, the amplitude of the spline increases significantly, inducing more uncertainty about the forward price level. This fact becomes visible in Figure 2.6 panel 1, where the HPFC shows continuously increasing oscillations on the yearly scale in the long end of the curve.

A drawback with the approach by Benth, Koekkebakker, and Ollmar, 2007 is that the number of parameters used in the fitting of the curve is dependent on the number of Futures observed. In Figure 2.5 panel 2 we have constructed two adjustment curves, one where the 3 first monthly Futures and 3 quarterly Futures are used as input. For the derivation of the second curve we took in addition the 4th monthly Future, leaving the other Futures prices used as input for the first curve unchanged. The result is shown in figure 2.5.

The result is that the curves will change which is in some sense counter-intuitive. This means actually that once a new maturity is becoming available in the market, this will change the PFC. This means that by adding Futures of other maturities will alternate the market expectation for all forward prices along the curve. This result shows a shortcoming of the method of Benth, Koekkebakker, and Ollmar, 2007

This becomes relevant when new maturities becomes available from one day to the other, which will lead to two different curves.
In both approaches, the seasonality shape is calibrated to historical spot prices, and it is exogenously inserted in the optimization problem. That means that the forecasted shape replicates the historical oscillations in prices. However, it has been empirically observed that the increasing in-feed from wind and photovoltaic in Germany has decreased the level of electricity prices over time (see Paraschiv, Erni, and Pietsch, 2014 and Paraschiv, Bunn, and Westgaard, 2016). In consequence, the traditional spreads between peak and off-peak power prices has been narrowed. Furthermore, due to the volatile renewable energies, in particular in case of very cold or hot years, the seasonality shape can no longer be considered as standard and it is more difficult to

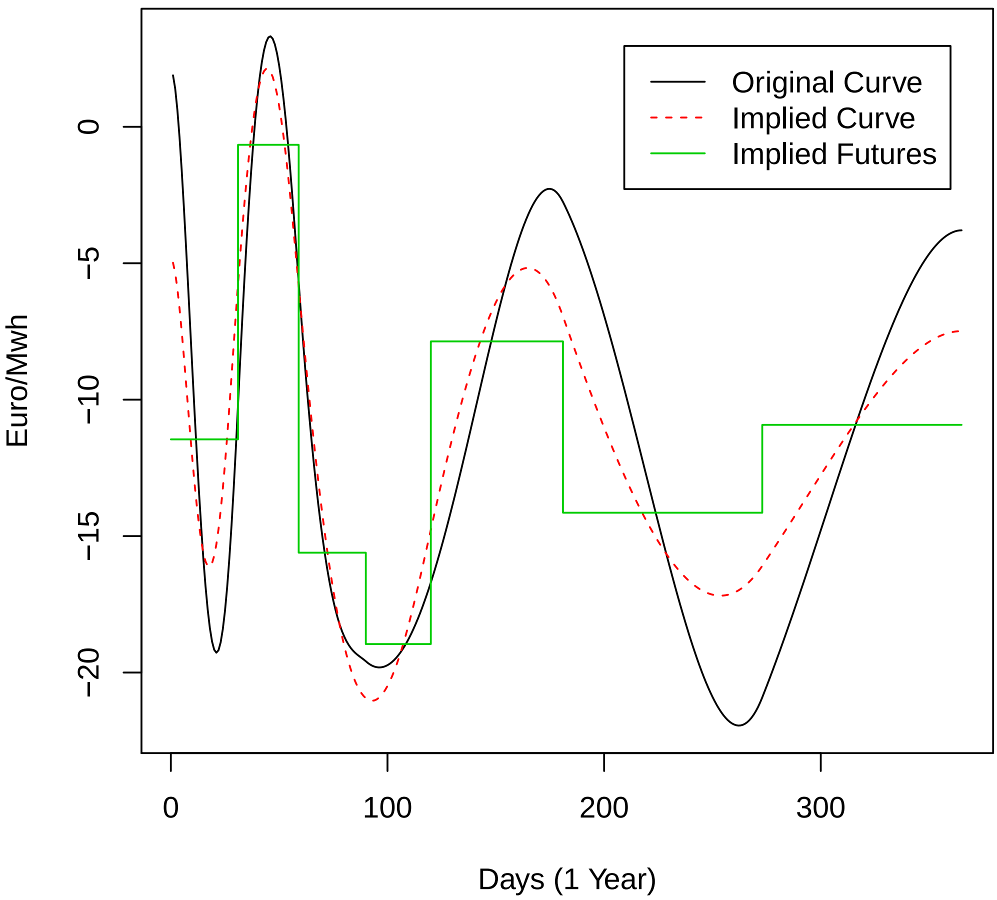

FIGURE 2.5: The adjustment functions generated based on the approach by Benth, Koekkebakker, and Ollmar, 2007, the black line is generated on the following input: The observed, de-seasonalized, prices for the first three monthly Futures and three quarterly Futures; the second curve has as additional input the 4th month Future. The straight lines are the corresponding Futures prices the second curve is fitted to.

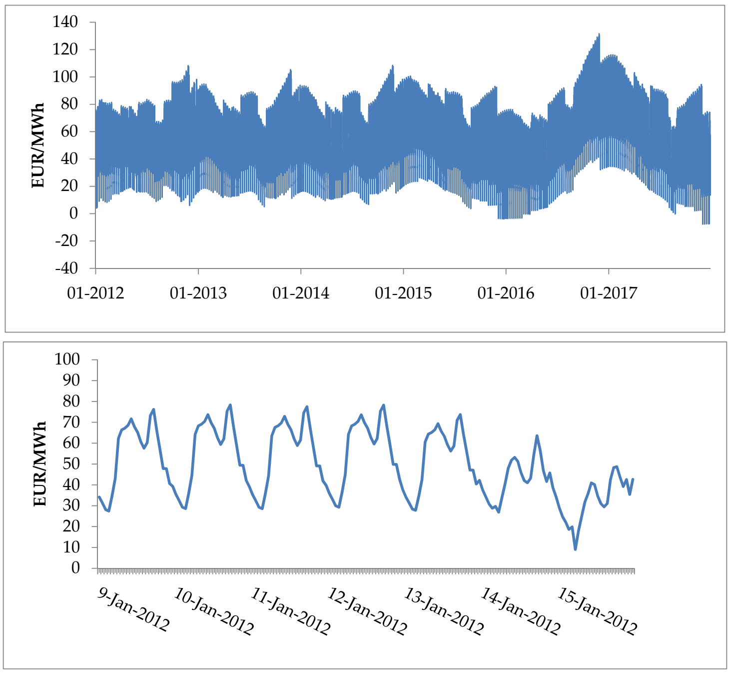

FIGURE 2.6: Example of one generated HPFC for the German PHELIX electricity index, based on the approach by Benth, Koekkebakker, and Ollmar, 2007 and having as input the observed Futures in the market at 1st of January 2012. In the lower panel we show one arbitrary week on the curve.

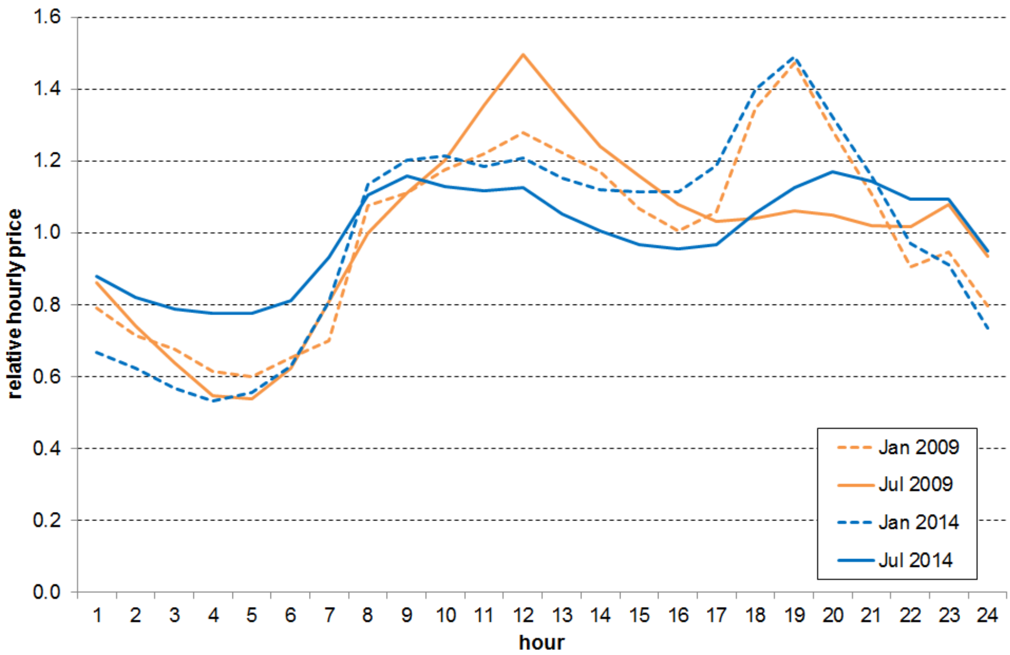

FIGURE 2.7: Change in the Daily Seasonality Shape of German Electricity Prices. The Graph shows the average prices of hourly contracts relative to the average base price for one month.

construct a correct seasonality shape. In this context, the reliance on pure historical prices for the derivation of the seasonality shape cannot realistically reflect the current dynamics.

As observed in Figure 2.7, the distribution of prices throughout the day has changed significantly during the month of July from 2009 to 2014, where the typical high midday peak has been decreased due to the increased in-feed of photovoltaic during the day. However, we do not observe the same for winter months. This example shows that the amplitude of the daily oscillation became smaller over time for summer months. Such typical changes are therefore not realistically reflected in the historically derived shape.

To overcome this methodological drawback, we propose a novel approach for the derivation of PFC's, where we allow the seasonality shape to reflect historical oscillations, but in the same time we adjust the amplitude to the observed Futures prices.

2.3.3 Novel Modeling Approach for PFCs

In this modeling approach we propose a joint optimization procedure where the seasonality shape is not treated exogenously, but it is simultaneously fitted to the historical spot prices and to the currently observed Futures prices. We believe that the amplitude of the oscillations along the seasonality curve should fit the market expectation about the level of the Futures prices with different delivery periods.

Mathematical specification of the novel model:

We model the seasonality curve and the correction term by one trigonometric spline, which is defined as follows:

$$\begin{aligned} f(t;m) = \quad & C + \sum_{i=1}^{6} \left[a_i \sin(\frac{2\pi i(t+S(m))}{12 \cdot M(m)}) + b_i \cos(\frac{2\pi i(t+S(m))}{12 \cdot M(m)}) \right] && (*) \\ & + a_4^{Q(m)} \sin(\frac{8\pi(t+S(m))}{12 \cdot M(m)}) + b_4^{Q(m)} \cos(\frac{8\pi(t+S(m))}{12 \cdot M(m)}) && (**) \\ & + a_{12}^{Q(m)} \sin(\frac{24\pi(t+S(m))}{12 \cdot M(m)}) + b_{12}^{Q(m)} \cos(\frac{24\pi(t+S(m))}{12 \cdot M(m)}) && (***) \end{aligned}$$

Here t is the time in days parameter, $1 \leq t \leq 365$, and m [2] is a parameter keeping of the months and $M(m)$ is the corresponding number of days in that month:

$$M(m) = \{\# \text{ days in month } m\}$$

for $1 \leq m \leq 12$. For example by choosing $m = 1$ (January), we get $M(1) = 31$.

The term $S(m)$ is chosen to ensure continuity of the curve between the transition times of the months. As an example, when going from January to February, we get $M(1) = 31$ and $M(2) = 28$, the transition between January and February takes place when $t = 31$. This means that for the curve to be continuous, we need that:

$$\frac{31 + S(1)}{31} = \frac{31 + S(2)}{28}$$

since we have 11 transition points between months, and 12 variables $S(i)$, we have one free variable, therefore we choose to set $S(1) = 0$. This gives us $S(2) = -3$.

Continuing in this framework we get that $S(m)$, must satisfy the following equation:

$$\frac{T(m) + S(m)}{M(m)} = m; \ 1 \leq m \leq 12$$

holds where the variable $T(m) = \sum_{j=1}^{m} M(j)$ counts the days from the first of January until the last day of the month m, then $T(1) = 31$, $T(2) = 59$ and so on, which are the time-points we are interested in.

Explanation of the different terms: The different parts of the function generating the PFC can be explained in this way:

The first part * will not differ significantly from a standard truncated Fourier series, but this choice of periodicity links the PFC to the months, and therefore to the Futures prices better.

As a standard truncated Fourier series can be too regular to correctly estimate the complex structure of electricity prices, we will add more flexibility by including a

[2] m is a function of time t, as $m(t) = 1$ for $1 \leq t \leq 31$, and $m(t) = 2$ for $31 < t \leq 59$, and so on, meaning we could skip the parameter m, and only use the parameter t.

spline trigonometric curve, by the terms in lines denoted by **and *** denoted by the superscript $Q(m)$ in equation ***. These parameters are allowed to vary across quarters. The terms in line ** will account for the flexibility of the curve, while the terms in line ***, ensure continuity and continuous derivatives. The fact that we have different parameters in the different quarters represents the spline part of the curve.

The constant C represents the mean level of the curve, while the other parts will describe how the prices distribute throughout the year. From now, on we will refer to the first term colored in black as the Fourier term, and the parts in red and blue as the spline terms.

Parameter selection:

The choice of the number of parameters in the Fourier term was determined by using Lasso regression trying to determine the number of significant factors. We started with 24 different terms and reduced it to 12, but there is still reason to believe that the number of relevant factors can be improved, especially by also changing the number of spline terms. This leads to the following set of parameters:

$$x = (a_1, ..., a_6, b_1, ..., b_6, a_4^1, ..., a_4^4, b_4^1, ..., b_4^4, a_{12}^1, ..., a_{12}^4, b_{12}^1, ..., b_{12}^4)$$

Fitting of the Curve:

The general idea behind the fitting procedure is if a class of functions shows a reasonable fit to observed historical seasonalities, then these functions should also be able to replicate the observed prices of traded Futures products. In our model we reflect the seasonality pattern of spot prices by the trigonometric functions introduced before and simultaneously fit align the seasonality curve to the observed Futures. Since our seasonality curve is linear in the parameters, this is the same as solving a constrained least squares optimization problem.

Our problem reads as follows:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - y\|_2 \\ \text{subject to} \quad & Cx = V \end{aligned}$$

where Ax, see appendix A for the specification of A, is our seasonality linear function and y represents the historical spot prices. In the constraints' matrix C, we will ensure the no arbitrage condition by ensuring that the PFC correctly replicates the observed Futures prices. As we are working with a trigonometric spline, the matrix C also needs to include the continuity constraints. A solution is obtained by solving the linear problem:

$$\begin{bmatrix} 2A^T \cdot A & C^T \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^T \cdot y \\ V \end{bmatrix}$$

If the matrix on the left-hand side is invertible, the optimal solution $\hat{x}$ is defined by:

$$\begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 2A^T \cdot A & C^T \\ C & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2A^T \cdot y \\ V \end{bmatrix}$$

It should be noted that if the solution x^{OLS} from the ordinary least squares problem, obtained by fitting the model to only the historical spot prices, already solves $C\hat{x}^{OLS} = V$, then these two solutions coincide.

The matrix C together with the vector V corresponds to the constraints and can be decomposed into two matrices:

$$C = \begin{pmatrix} H \\ G \end{pmatrix}$$

where H corresponds to the Futures and G corresponds to the constraints needed on the spline part of the curve. The solution to our optimization problem x^* gives us the desired price forward curve $f(t)$, which is computed by the matrix multiplication Ax^*. The curve here does not include a weekly or daily seasonality, and is therefore meant to describe the distribution of the prices throughout the year. The weekly and daily seasonalities can be included by methods described earlier in this section.

This approach depends on the fact that we use a seasonality function that is linear in the parameters. However, it is flexible enough that one can use the same method by taking a seasonality curve based on the standard Fourier series, dummy variables or some other class of functions that are linear in the parameters.

Downsides with the novel modeling approach:
As argued for earlier, as you expect the evolution between normal days to be smooth, there are periods throughout a year when one does not expect smooth transition, typically when going to and from holiday periods, these characteristics are hard to model with a smooth curve and should be taken care of in an ad hoc step.

The model presented also does not include a term designed for taking care of the weekly seasonality, so this curve represents how the prices are distributed throughout the year, excluding the weekly pattern. In our estimation results we will use a weekly seasonality component modeled by dummy variables, as in Paraschiv, Fleten, and Schürle, 2015. One can either add the dummy variables directly in the optimization method, or one can add a weekly seasonality after the optimization.

2.4 Estimation Results

In this section, we will asses comparatively the performance of the various methods discussed in this study to generate PFCs. We assume that every forward price of a certain maturity along the PFCs should meet in expectation the realized spot price. We are aware that there are deviations between the price forward curve and the realized spot prices due to the risk premium component. We do expect our risk premium to be the same for all price forward curves, and the therefore we believe our criteria of comparison for PFCs is realistic.

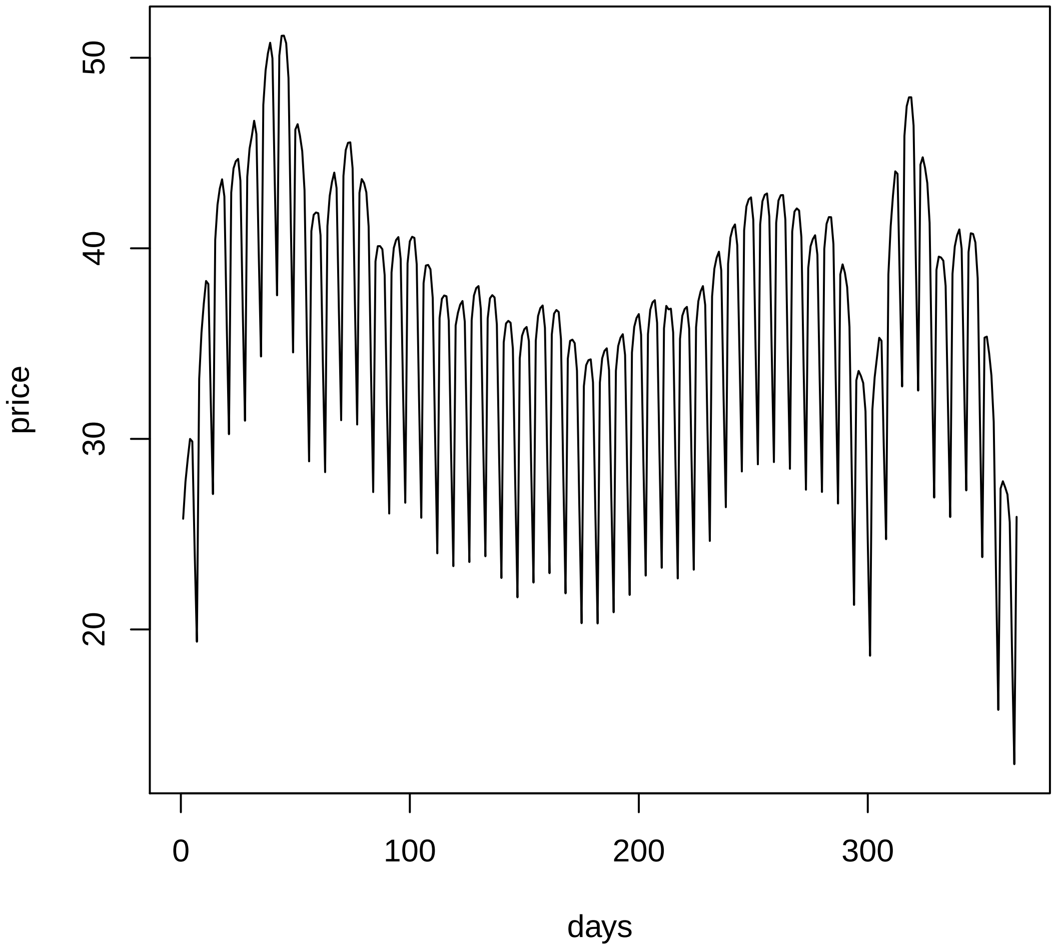

FIGURE 2.8: Result of a PFC for 2015 made of a trigonometric spline to model the yearly seasonality and dummy variables to model the weekly seasonality

2.4.1 Data Used

We have generated PFCs based on four different methods. To test the validity of the curves, we constructed two sets of HPFCs, each including three curves, based on the methods discussed in this study: Fleten and Lemming, 2003, Benth, Koekkebakker, and Ollmar, 2007 and our novel approach. One fourth curve was generated based on Fleten and Lemming, 2003 where we added ex-post the daily and weekly seasonal pattern from Equation (2.1). The reason is that the standard approach of Fleten and Lemming, 2003 suppresses the weekly and daily seasonal patterns if we include the smoothness. All curves are generated for the year 2015. The first set of curves are estimated based on historical spot prices from 2011-2013 used to fit the seasonality curve and on Futures products observed in 2014 for 2015. The Futures prices used cover the first three months, and the three following quarters. This will be our out-of-sample analysis. The second set of curves will be our benchmark, they were constructed by taking the observed spot prices for 2015 and as a proxy for the Futures we took average of the realized spot prices over each month for the corresponding delivery period. This will be our in-sample analysis. For the methods by Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 we will use a seasonality curve based on dummy variables, as described in Paraschiv, Fleten, and Schürle, 2015. Our novel approach is described in section 2.3.3, due to the technical specification of our model we can not take the dummy based seasonality shape in the comparative assessment of the produced HPFCs.

2.4.2 Comparative assessment of generated price forward curves

The set of curves have been generated for a weekly daily and hourly resolution and then compared to average observed weekly, daily and hourly spot prices.

In tables 2.2 and 2.3 we show for each method the in- and out of sample performance as:

$$|EstimatedPriceWeek_w - RealPriceWeek_w|$$

Where $EstimatedPriceWeek_w$ is the generated price from the PFC and $RealPriceWeek_w$ is the observed average spot price for the corresponding week.

As seen in Table 2.2, the novel modeling approach scores best for 33 out of 52 weeks, while the other methods score best for 15 and 4 weeks, respectively. This comes from the fact that the methods by Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 are relatively flat during one month, as observed in figure 2.9 (the price for the estimated weekly forward prices are constant within one month), while the novel approach allows for more variability during the course of one month. This comes from the fact that the novel modeling approach uses continuous functions as a basis for the seasonality curve instead of dummy variables. Thus, our novel approach is more parameter intensive, which helps to shape better the curve. However, this feature might lead to over-fitting, which can explain why our model performs better in-sample, but it loses accuracy out of sample as we observe in 2.3. Indeed, when we go out of sample we observe an overall increase in the deviations between the observed average weekly spot prices and the estimated prices for all models. The increase in the errors in the out of sample case study shows that historical data are not a good enough estimator of the future market expectations.

Week	Novel	Fleten	Benth
1	**1.34**	5.78	5.89
2	**5.52**	8.41	8.45
3	**3.69**	7.19	7.18
4	**0.98**	6.18	6.13
5	4.63	**3.75**	3.92
6	**0.65**	2.17	1.93
7	**1.80**	1.84	2.02
8	3.68	2.20	**2.05**
9	**3.00**	5.37	5.25
10	5.16	**3.02**	3.11
11	**1.33**	2.24	2.15
12	**3.15**	5.38	5.05
13	**7.19**	7.48	7.86
14	1.77	**1.24**	1.79
15	**2.73**	3.61	3.32
16	**0.69**	1.73	1.76
17	**0.94**	1.70	1.48
18	**0.45**	0.80	0.84
19	**0.27**	1.95	2.09
20	1.20	**0.13**	0.56
21	**1.78**	2.60	1.88
22	**1.18**	1.86	2.20
23	1.90	1.77	**1.44**
24	2.42	**0.92**	1.08
25	2.55	**0.17**	0.23
26	2.12	**1.76**	1.80

Week	Novel	Fleten	Benth
27	**1.43**	1.85	2.32
28	1.34	**0.00**	0.79
29	**4.30**	5.53	5.56
30	**3.37**	5.33	4.31
31	**3.53**	4.37	3.66
32	**0.22**	2.66	3.21
33	**0.15**	3.67	3.80
34	**1.95**	3.77	4.66
35	8.35	1.33	**0.22**
36	**0.69**	3.78	4.06
37	1.05	**0.38**	0.41
38	1.66	**1.31**	1.51
39	1.66	**1.49**	2.32
40	**1.53**	3.31	3.96
41	1.95	**1.34**	1.35
42	**2.96**	4.25	4.51
43	1.70	1.58	**1.00**
44	**3.67**	5.07	4.96
45	2.15	**0.80**	1.31
46	**3.69**	4.50	4.72
47	4.50	**2.81**	2.84
48	1.40	**1.13**	1.26
49	**1.38**	4.33	4.17
50	**3.33**	5.30	5.36
51	**2.62**	3.05	2.83
52	**3.59**	7.10	6.75

TABLE 2.2: Show the absolute mean of estimated week mean - real week mean for week 1 to 52, as calculated from in-sample data from 2015

In the following we will continue with more statistical tests for daily and hourly prices.

We compare further the performance of the fours PFCs based on the following statistics: We computed the absolute, the squared error and the Mean Average Percentage Error (MAPE). The results are available in table 2.4.

$$AbsoluteError = \frac{1}{n}\sum_{i=1}^{n}|RealizedPrice_i - EstimatedPrice_i| \tag{2.16}$$

$$SquaredError = \frac{1}{n}\sum_{i=1}^{n}(RealizedPrice_i - EstimatedPrice_i)^2 \tag{2.17}$$

$$MAPE = \frac{100\%}{n}\sum_{i=1}^{n}\frac{|RealizedPrice_i - EstimatedPrice_i|}{|RealizedPrice_i|} \tag{2.18}$$

the novel modeling approach scores best for all the in sample tests, while the

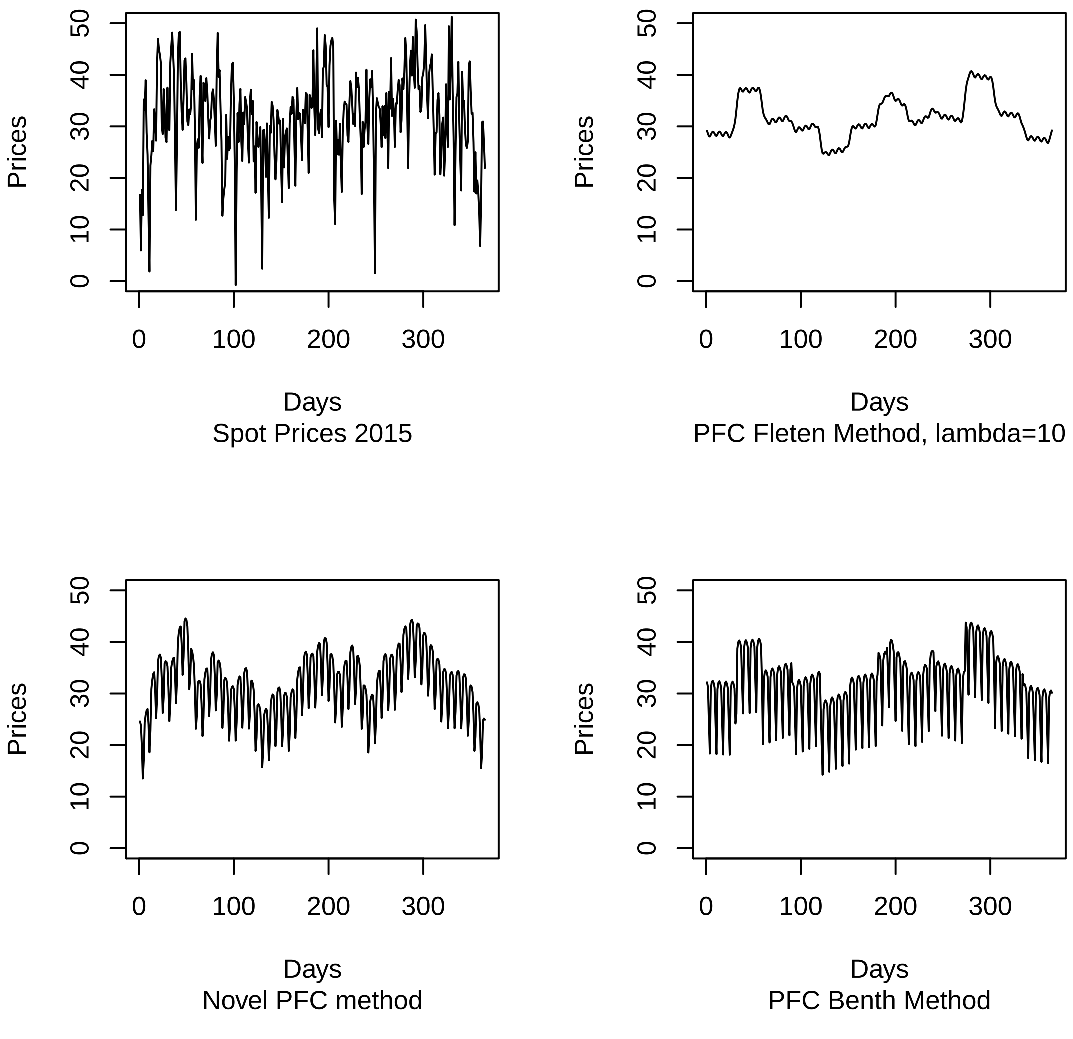

FIGURE 2.9: The graph in the top left panel show the evolution of the spot prices used for the in sample calibration. The three other graphs represents the PFCs generated based on the three different methodologies.

Week	Novel	Fleten	Benth	**Week**	Novel	Fleten	Benth
1	17.69	8.25	**4.79**	27	18.98	**15.97**	19.31
2	7.23	6.29	**4.88**	28	19.49	**16.63**	19.50
3	**4.79**	9.32	10.46	29	14.84	**12.64**	14.66
4	**11.08**	15.63	18.59	30	**9.14**	9.25	12.34
5	**8.97**	11.69	12.96	31	14.14	**14.09**	16.31
6	**0.67**	3.83	3.67	32	8.64	**8.12**	9.68
7	**8.56**	10.61	10.86	33	15.34	**14.91**	15.37
8	18.15	**14.24**	14.96	34	19.52	19.33	**17.69**
9	23.30	**18.62**	19.28	35	15.88	17.10	**14.97**
10	17.45	**17.03**	17.33	36	13.24	15.10	**12.36**
11	**17.82**	19.96	20.01	37	12.57	14.53	**11.43**
12	**17.07**	18.18	17.71	38	16.73	18.99	**15.53**
13	21.05	20.15	**18.90**	39	16.12	18.32	**14.84**
14	19.39	20.20	**17.66**	40	**10.69**	11.42	12.03
15	17.04	18.43	**15.72**	41	**12.23**	13.91	14.54
16	21.72	21.89	**19.22**	42	16.65	**14.98**	15.33
17	14.94	13.54	**10.98**	43	20.67	**14.71**	14.78
18	20.62	20.32	**19.05**	44	17.45	13.74	**12.94**
19	**25.91**	27.12	26.33	45	**10.37**	14.51	13.00
20	**25.63**	26.69	26.39	46	**11.73**	18.92	17.12
21	25.07	**24.40**	24.65	47	**17.14**	19.55	17.56
22	23.25	**22.20**	23.52	48	13.31	13.35	**12.15**
23	**25.57**	25.93	28.50	49	**6.19**	10.36	11.84
24	**24.73**	25.37	28.36	50	**8.00**	11.60	12.91
25	14.98	**14.38**	17.59	51	11.94	**9.01**	10.20
26	20.39	**18.45**	21.83	52	6.79	1.79	**0.50**

TABLE 2.3: Show the absolute mean of estimated week mean - real week mean for week 1 to 52, as calculated from out-of-sample data

second Fleten method scores best for the out of sample tests. This result can be related to the differences in the technical specifications of the models: In our approach the over-fitting property of the seasonality function applied to historical prices leads in this case to miss-estimation of the future price level. In Fletens approach where we used the exogenous defined seasonality shape based on dummys we get a more rough approximation of the historical seasonality, which leads to a slightly better out-of-sample fit. In any case, there are no major differences for the different methods in the in- and out of sample results.

2.5 Conclusion

In this study we make a comparative study of how different frameworks used in the construction of the HPFC compare to each other. We compare different methods for the seasonality function, adjustment function and how the smoothing is done in the different models. The methods investigated in this model is the adjustment functions proposed in Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 combined with the seasonality function based on dummy variables from Paraschiv, Fleten, and Schürle, 2015 as well as a novel approach where we do a combined fitting of the seasonality curve and the adjustment function on a curve based

Daily Scale					
Test	Data	Benth	Novel	Fleten 1	Fleten 2
MAPE	In sample	32%	**29%**	45%	32%
MAPE	Out of sample	67%	42%	57%	**41%**
Absolute Difference	In sample	4.79	**4.57**	6.05	4.85
Absolute Difference	Out of sample	8.76	6.79	7.15	**6.32**
Hourly Scale					
Test	Data	Benth	Novel	Fleten 1	Fleten 2
Absolute Difference	In sample	5.95	**5.83**	7.15	5.98
Absolute Difference	Out of sample	9.92	8.09	8.55	**7.67**
Square Difference	In sample	65.71	**61.69**	91.46	65.89
Square Difference	Out of sample	181.69	116.86	139.76	**109.25**

TABLE 2.4: Comparison of the different models to the realized spot prices. Fleten 1 is the original Fleten method, while Fleten 2 is where we have reapplied the weekly seasonality

on trigonometric splines.

These methods all have their strength and weaknesses, and we conclude that the important thing is to understand the characteristics of the different models, and how these can be used to construct a HPFC fitting ones individual beliefs. We will here give a short summary of the strengths and weaknesses of the different approaches, both for the seasonality curve and the adjustment function.

Dummy variables vs. Functional form: A seasonality function based on dummy variables do not allow for a continuous curve, resulting in large price jumps between periods modeled by different dummy variables, typically months, which is an undesirable feature of the HPFC. A functional form for the seasonality curve, like the trigonometric spline described earlier, has the opposite problem, not being able to model sudden price movements which is the case when moving from week to weekend, or between individual hours when new power plants are taken in/out of the production-mix to cope with peak/off-peak hours.

As a conclusion one should either smooth the curve based on dummy variables, while taking care not to suppress the weekly/daily seasonality, or if using a functional model, individually model periods where one typically observes sudden price movements, in an ad hoc manner.

Fleten Method: In the method by Fleten, one simultaneously smooths the seasonality curve while fitting the curve to Futures prices. The smoothing of the seasonality curve is reasonable when one is not satisfied with the smoothness of the seasonality curve, but one should be cautious to not suppress the weekly/daily seasonality, as shown earlier. In our tests, the method by Fleten seems to perform the best, out of all models, for the out-of-sample testing if one ad hoc reapplies the weekly/daily seasonality pattern. When we reapply this pattern it also scores in all tests better than the original Fleten model where this pattern is not reapplied after the smoothing of the seasonality curve is done.

Benth Method: The method by Benth uses a polynomial spline of the fourth degree to model the adjustment function. We have discussed two downsides with this method: One is that the smoothing is only done on the adjustment function, which is positive if one is already satisfied with the smoothness of the seasonality curve, as one is not suppressing any weekend/daily seasonality, but not suitable if one wants to smooth parts of the seasonality curve. The other downside is that the number of parameters are dependent on the number of Futures products observed, resulting in a deterministic change of the curve when new products are added to the market, which can be used to form arbitrage strategies.

Novel Method: The novel method is based on a constrained least square optimization procedure, where the underlying function is a trigonometric spline. We observe this method is the best for replicating the spot prices in an in-sample test, but does not outperform the other models in the out-of-sample tests. We attribute this to two reasons: In this framework we need more variables for the seasonality curve, as we want it to be able to replicate the observed Futures prices. When we do not observe all Futures products, we then obtain more free variables, leading to an over-fitting of the curve. Secondly this method allows for more variability in one month, resulting in different prices for the different weeks in one month, which is less so the case for a curve based on dummy variables. We conclude that since this curve does not perform any better for the out-of-sample testing, the patterns of which week during one month has the highest price, is not necessarily reoccurring for subsequent years.

This method can perform well if one has specific variables in the seasonality curve linked to a specific period. If the curve has dummy variables for the months, these could be set to match the price of the corresponding monthly Future, but one should not include too many variables to match the whole set of Futures products, as this can lead to over-fitting.

Chapter 3

Dynamics of the PFC

3.1 Introduction and Layout of Section

3.1.1 Introduction

In the literature several methods are suggested for constructing what we in this thesis call the adjustment function of the PFC. Most of these studies focus on having a mathematically tractable model, where they want to minimize some distance measure, typically measuring the smoothness of the resulting PFC. In this thesis we will instead consider what economical features are natural for an adjustment function. Most models used for the adjustment function in the literature comes originally from interest rate modeling or other branches of economics, and therefore not originally constructed to be economically viable for PFCs for electricity.

The methods we will discuss are the methods by Fleten and Lemming, 2003, which in turn is the Hodrick-Prescott filter first introduced in Hodrick and Prescott, 1997, used for smoothing out time series, with an additional constraint ensuring the PFC is arbitrage free to the observed Futures prices. We will also discuss the method suggested by Benth, Koekkebakker, and Ollmar, 2007, which origins from Adams and Van Deventer, 1994 as a way to fit yield and forward rate curves with maximum smoothness. A third approach was recently published by Caldana, Fusai, and Roncoroni, 2017 where they use a Monotone Convex Interpolator first used in Hagan and West, 2006. We will also discuss our novel method which we have explained earlier, based on a constrained least squares approach used on trigonometric spline functions. In the following we will frequently refer to the methods in Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007, when not refering to the paper, we will denote them by the Fleten or Benth model respectively.

There are also methods that do not come from other fields of study, but are solely used for the construction of PFCs for electricity prices. We will look at the method used in Biegler-König and Pilz, 2015 which was first described in Burger, Graeber, and Schindlmayr, 2007 where they shift the seasonality curve to the level of the Futures prices. We have described three of these methods in detail earlier, we will briefly describe the two others later. We only implement the methods by Fleten, Benth and the Novel method, so when we analyze the methods with respect to data we will focus on these methods.

In this section we will discuss the following problem: For a given seasonality curve, or for a constructed PFC, what happens when we observe that this object does not replicate the observed Futures price. Either because the prices of the Futures products have changed form when we estimated our curve, or because we did not adjust the seasonality curve to the observed Futures prices yet. We will explain how this shift of the curve is done in the different models, first by observing how the constructed PFCs change when the Futures prices change.Secondly we will find an analytical solution for the derivative of the PFC with respect to the observed Futures prices. By doing this we will figure out which of the characteristics of these methods seems reasonable for an optimal model for the adjustment function.

This question is not straight forward to answer as we do not have data backing up the claims we will present. This is because constructed PFCs from retailers of electricity are not publicized, meaning we do not know what a typical response to changes in the Futures products are, as seen from a trader of electricity. We therefore need to discuss what seems natural from a logical perspective. We will simplify our reasoning by only focusing on what should happen when the price of one Futures product changes. For example, if the Futures price for June increase with 1 €/MWh, one knows the total change in the day prices for June needs to change with 30 €/MWh as there are 30 days in June, and that the price change in all other months needs to remain constant. Natural questions concerning this are:

1: Should the prices throughout the whole of June increase?
2: Should the current level of the June Future and the other observed Futures products affect this change?
3: Should the current price of each single day affect this? For example should the Monday-Friday prices change more than the weekend prices, as these prices are typically higher.
4: Should the prices for days not in June be affected? For example, if the price for the 30th of June increase, a natural assumption is that the price for the 1st of July will increase as well. This consequently mean that the price of some day in July needs to decrease to keep the no-arbitrage condition for July. The effect a given Futures product has on prices not covered by this product will in the following be called the spillover effect.
5: Should the number of Futures products observed affect this change in the curve? For example, should this be dependent on whether we can trade in only the 3rd quarter Future or in the individual months covering this quarter as well.

The goal of this section is to investigate how the PFC changes with respect to changing Futures prices, and changing granularity of observed Futures products. From this we will conclude what characteristics are natural, and which are not. We will start our study by investigating plots which show how the estimated price for one day changes when the input used to estimate the PFC changes as it would have done during one year. By doing this we will observe typical price-developments of the PFC with respect to the data used for the fitting. By observing the differences in the different models we get a picture of what is a natural way for the PFC to evolve, as time-to-maturity becomes shorter.

Afterward we will discuss plots where we observe how the PFC change when only the price of one Futures product change. This will not be a realistic market occurrence, as the prices of the observed Futures products all change continuously in time, but by doing this we can easier observe how the relationship is between the Futures products and the resulting PFC. We will also observe how the different curves reacts to the cascading of products into products with smaller delivery periods.

In this section we assume that the seasonality curve is remaining constant, and that only the adjustment function change in time. This is based on Biegler-König and Pilz, 2015, where they argue for why the seasonality shape is only updated infrequently, and that the adjustment curve (which they call shifting) is the only thing that is changed on a day to day basis. Because of this, we will focus on what happens when this is done in the following, and keep the seasonality curve constant. Biegler-König and Pilz, 2015 also present a new way of adjusting the seasonality curve to the price level of the observed Futures, which is based on the method proposed in Burger, Graeber, and Schindlmayr, 2007, we will discuss this method later. We will refer to the paper by Biegler-König and Pilz, 2015, instead of the book Burger, Graeber, and Schindlmayr, 2007, when referring to this model, as the paper is easier available than the book.

Benth and Paraschiv, 2017 construct a unique set of 2386 HPFCs for PHELIX, the German electricity index, between 01/01/2009 and 15/07/2015, each with a duration of 5 years. These curves are then truncated and the first two years of each curve is analyzed. The construction of the curves is done using the seasonality curve from Blöchlinger, 2008 and the adjustment function from Fleten and Lemming, 2003, with $\lambda = 0$, as we have described earlier. They consequently observe the output from this as a random field and analyze this set PFCs, which in-turn lead to a spatial-temporal model for the forward prices. This analysis is in theory only an analysis of a transformation of the Futures prices data, as they study the change in their HPFCs as these Futures prices change in time. Our goal in this section is to observe how this transformation of the Futures prices into PFCs are in the models proposed earlier, and from this draw conclusions on what is a correct way to transform this data.

Caldana, Fusai, and Roncoroni, 2017 compare how a shock on the Futures prices manifests in the HPFC constructed by two different methods. Both with a seasonality curve constructed by trigonometric functions, but where the adjustment function is either constructed by the Maximum Smoothness Interpolation (MSI) method, which is the before-mentioned Benth method proposed in Benth, Koekkebakker, and Ollmar, 2007. They also consider a method based on a Monotone Convex Interpolation (MCI), which is first used in Hagan and West, 2006 for yield curves. They conclude that the MSI method is significantly more volatile with respect to changes in the Futures prices than the MCI method. In their study they considered a curve constructed from real data observed at February 28, 2013, and how an artificial shock of -20 Euro/MWh at the shortest end in the time-to-maturity spectrum affects this curve. Our goal is to see if we can find a general rule of how the PFC will change, as a function of the Futures prices, in some sense the derivative of the PFC with respect to the Futures prices.

3.1.2 Layout of Section

As pointed out earlier, in this section we will study how the adjustment function affects the final PFC in the three previously discussed methods. From this we will draw conclusions on how an optimal adjustment function should be. We start our study by investigating two sets of plots. In Figure 3.1 at page 37 we observe how the prices change when we increase the price of the March Futures prices by a small amount. In Figure 3.2 at page 38 we observe how the estimated price for the 30th of June changes as time-to-maturity is decreasing. We have constructed the data set as follows:

1: First create a list of the 252 trading days of the year with the corresponding number of products traded at each day, as seen in Table 3.1 at page 60.
2: For each trading day we create an arbitrage free list of non-overlapping Futures products from the observed Futures products.
3: For this set of trading days and Futures products we construct a set of 252 PFCs for each of the three considered methods.

This study is meant to focus on what happens when we have a small random change to our Futures prices, what will an appropriate response of the PFC be to this change. This study is not meant to reflect what an appropriate response of the PFC is when we receive new market information, as in this case the change in the PFC should reflect this change specifically. For example, if there at on point comes information that a new photovoltaic power plant will be finished in three years time, this will drive the Futures prices from that year down. An automatic change of the PFC will drive prices down for the whole year, while a more natural response would be to only shift the prices for the summer months, especially for the day-light hours. Our research focuses on what happens when the prices change on a day to day basis, without new information in the market driving these changes, but random shocks as a result of trading of the observed Futures products.

The main questions we want to answer in this chapter can be divided into four parts:
1: How does the models suggested take into account changes in the observed Futures prices.
2: How does the models suggested take into account the introduction of a new traded Futures products.
3: How does the number of contracts, especially yearly contracts in the long end of curve affect the PFCs.
4: How can this information help us in the construction of an optimal way to model the adjustment function.

The rest of this section of the thesis is organized as follows: In Section 3.2 we give an overview of the data used in our analysis, we also present some initial plots motivating our research. In Section 3.3 we give a short review of the methods used, both the three models discussed earlier and the models from Caldana, Fusai, and Roncoroni, 2017 and Biegler-König and Pilz, 2015, as well as similarities and differences between these models. In Section 3.4 we introduce the derivative of the PFC with respect to our Futures prices. In Section 3.5 we discuss the differences in the adjustment function with respect to this derivative, and from this propose how to construct a new optimal adjustment function. In Section 3.6 we conclude.

3.2 Time Dynamics of the PFCs

3.2.1 Data used and a description of the electricity market mechanism

To successfully model how the PFC evolves in time,one needs to understand the market for trading electricity works, and how the different contracts are traded. As

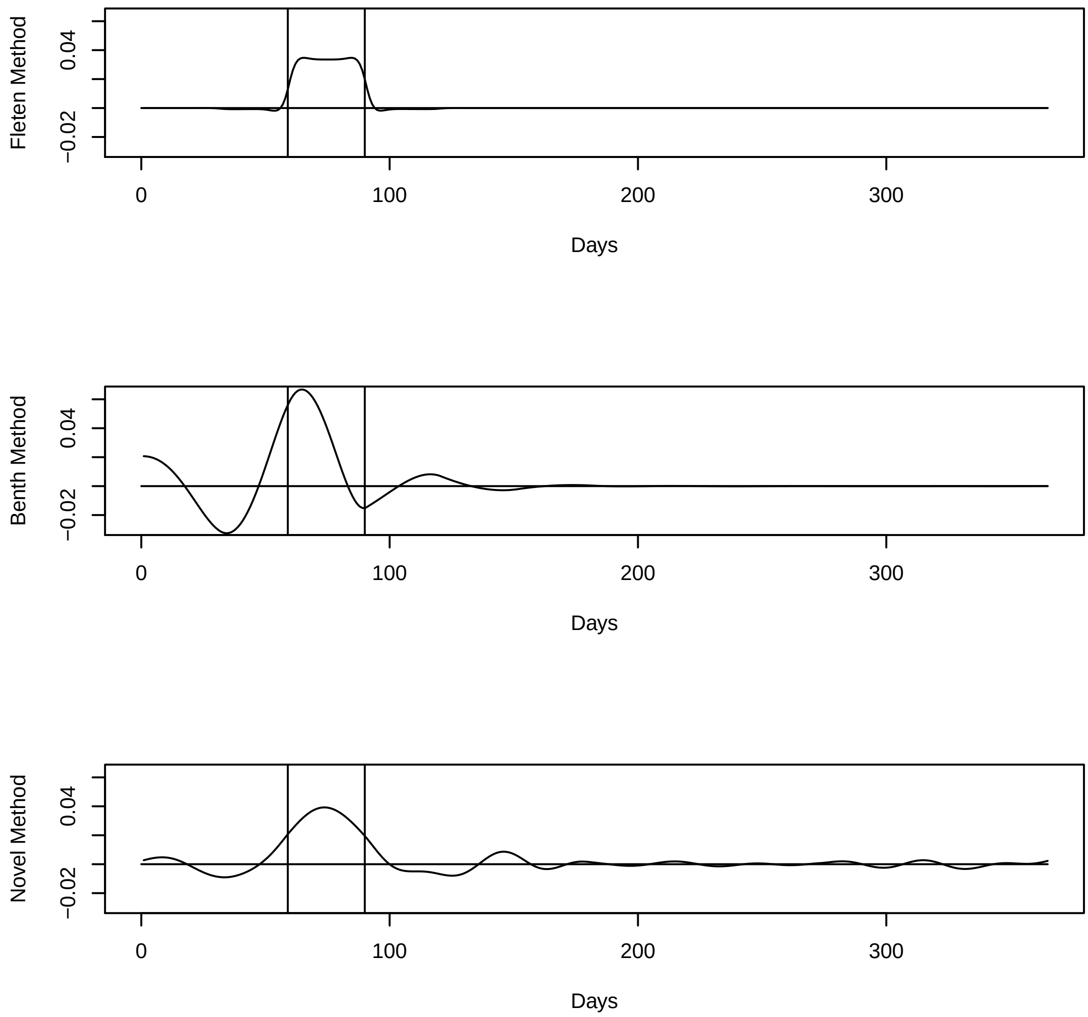

FIGURE 3.1: Artificial example where we show how the prices change when the price of the March Future change, assuming we have a Futures product covering each month. The vertical lines represent the start and end of March, while the horizontal line shows the null line. If the curve is over that line, the price is increasing with respect to the march Future, and under means the price is decreasing.

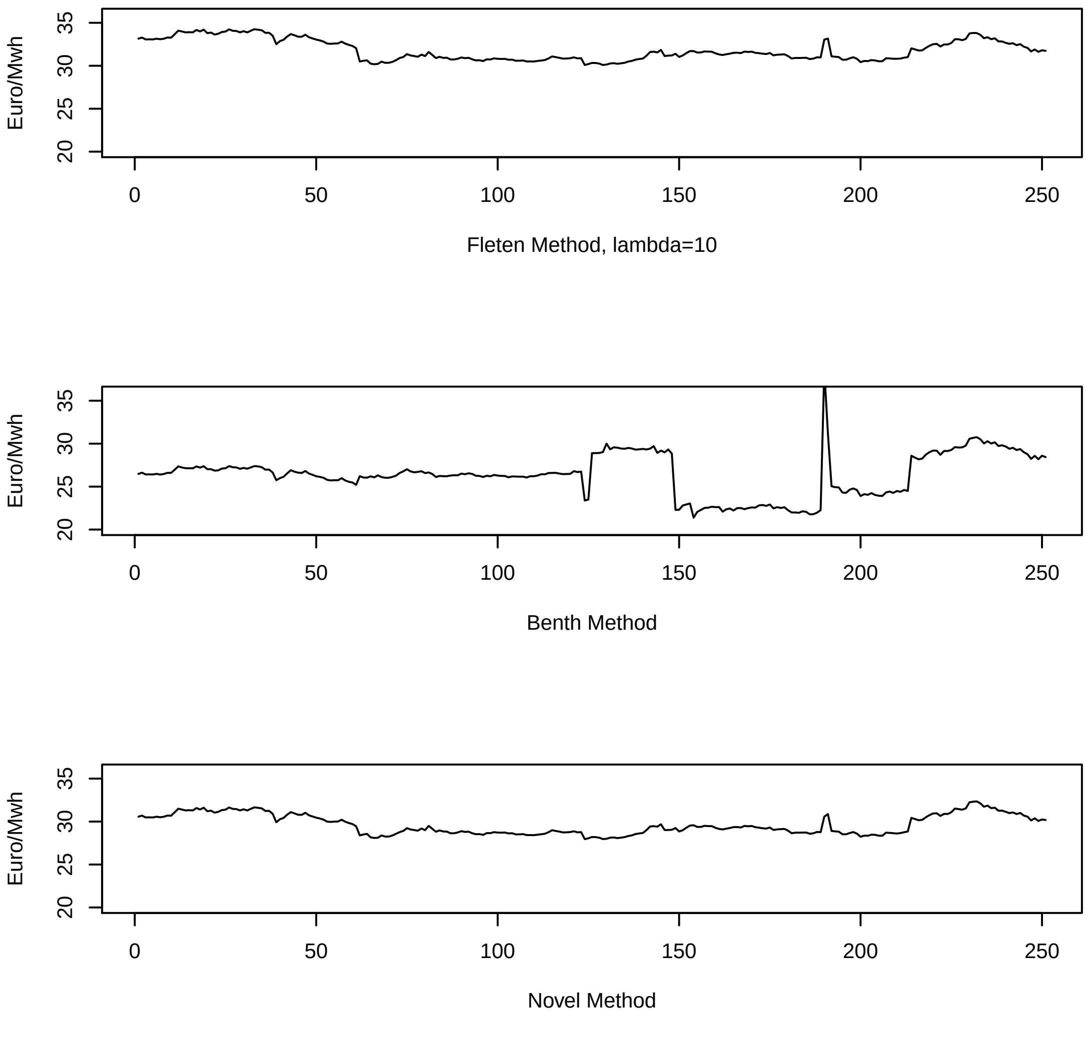

FIGURE 3.2: Estimated price for the 30th of June 2015, as seen from each trading day in 2014.

mentioned earlier we assume a simplified version of the market, where we assume only the change in the Futures products affect the PFC. As argued for in Biegler-König and Pilz, 2015, this is reasonable, as the seasonality curve is only infrequently updated, and we can consider our framework correct for the periods where the seasonality curve is remained unchanged.

In this thesis we have used historical spot prices from 2011-2013 to estimate the seasonality curve and then adjust the curves to the Futures prices observed in 2014 for the year 2015. As the main goal of this study is to understand how the Futures prices affect the PFC, it is vital to understand how these are traded at the EEX.

According to the EEX (European Energy Exchange), the maximum number of futures traded at the EEX is: The next 34 days, the next 5 week- ends, the current and next 4 weeks, the current and next 9 nine months, the next 11 full quarters, the next 6 full seasons and the next 6 yearly Futures. This is the maximum number of Futures one can see traded at any given day.As not all products are liquidly traded we haven chosen to take a subset of these products as the basis for our study. We have also chosen to take the ex-post approach, where we fit the PFC to last traded prices, and not the bid-ask spread. This is done since we want to have one specific PFC for each day to simplify the effect the change in the Futures prices has on the PFC. When looking at a bid-ask spread, all PFCs in this bid-ask spread would be viable.

The set of Futures we have chosen to work with is this, all: The current, and the six next months, the current and the next three quarters, as well as one Futures product covering the whole of 2015. This means in January 2014 we observe a Futures product covering the first quarter of 2015 and one Futures product covering the whole of 2015. The first occurrence of a monthly product is in July, and from that point one new monthly product for 2015 is added in each month. With these assumptions we will in August 2014 observe the monthly Futures products covering January and February, as we already observe the quarterly product for January-March we get an implied price for the monthly product covering the month of March from the no-arbitrage condition.

The number of Futures products used could be chosen otherwise, as more products are traded on a given day, but we have chosen this subset of the traded Futures since these are in general liquidly traded, which makes it easier to obtain a consistent framework for our analysis.

The monthly future is traded until the last trading day of that month, while the Quarter/Season/Year Futures has last trading day on the third exchange trading day before the beginning of the delivery period. From this we get that new monthly products are added on trading day number:

$$1, 22, 42, 63, 83, 104, 125, 148, 169, 191, 213, 234,$$

and quarterly products are added on trading day number

$$1, 61, 123, 189.$$

The total number of trading days in one year is 252 and we construct as many PFCs. In table 3.1 at page 60 we have listed the set of traded products available at the different trading days.

3.2.2 Initial Analysis of the Dynamics of the PFCs

In Figure 3.1 at page 37 we see how the prices change throughout the year when the March Futures price increase, while all other prices remain constant. From these plots we observe the following characteristics of the proposed models:

Fleten Method: The prices during the whole of March increase, and the prices quickly drops toward zero on both sides, giving a slight spillover effect for February and April, but virtual no spillover effect for the other months. The price seems to vary the same for most prices, apart from at the end-/start-points of February, March and April.

Benth Method: The curve shows the counter-intuitive characteristic that during the end of March the prices decrease even if the average price needs to increase. We also see a substantial spillover effect, meaning that a change in the March Future gives an effect on the prices for the whole year. This will also be the case if the period covered by the PFC is longer, a change in any Futures product will have an impact on the whole curve.

Novel Method: Unlike in the Benth method, the Novel method has a positive price increase during the whole of March when the price of the March Futures product increase. As in the Benth method we also see a great spillover effect, affecting the prices throughout the whole year.

A natural question is how does the price level of the observed Futures products affect this change. Will a price change from 25 to 30 €/MWh affect the prices differently than a change from 30 to 35 €/MWh, and is this change also dependent on what the prices of the other Futures products are. We will later show that the answer to both of these questions is no, meaning the price of a given day or hour in the PFC is linear in the observed Futures products. This consequently mean that the change in the PFC is independent of the seasonality curve in these three models when the number of Futures products remain constant. How the curve reacts to the inclusion of a new Futures product will however be dependent on the seasonality curve.

In Figure 3.2 at page 38 we have estimated the price of electricity with delivery on the 30th of June following the closing prices of the Futures products observed each trading day in 2014. This results in 252 different prices for the 30th of June for the Fleten, Benth and Novel model respectively. When changing the seasonality curve the only change will be the starting point and the jump size when a new Futures product is introduced, therefore between the points where new products are introduced, this can be seen as a the stochastic process following a certain dynamic, but where the starting point is dependent on the seasonality curve.

All these plots seems fairly different, and in this chapter we will explain these differences and similarities in the proposed models. The method from Fleten and the Novel method are both fairly stable compared to the method from Benth. The method from Benth is in general more unstable, and we observe more pronounced jumps when new Futures products are added to the market than in the two other models. Our the question is, which of these characteristics are natural for the price development and which should be rejected. In the following we will show a mathematical relationship between the Futures prices and the corresponding PFC which will give insight in what is a natural model for the adjustment function.

3.3 A review of modeling approaches for price forward curves

3.3.1 Approaches for modeling the adjustment function

In the current study, we will focus our attention on three different approaches for the derivation of the adjustment function, namely Fleten and Lemming, 2003, Benth, Koekkebakker, and Ollmar, 2007 and our Novel method. In the two first studies the seasonality shapes have been historically derived and represent an exogenous input for the derivation of the price forward curves, while in the third study the seasonality shape is derived simultaneously with the fitting to the observed Futures products. The two first optimization procedures have as a main objective the minimization of the distance between the seasonality curve and the resulting price forward curve, under certain constraints. The Novel method has as a main objective to minimize the distance between the PFC and historical prices, given that the PFC replicates the observed Futures prices. The curve should be arbitrage free and the constraints ensure that the average of the forward prices on the different segments on one curve meet the corresponding level of the observed Futures prices.

We will also briefly discuss the methods proposed in Hagan and West, 2006 and Biegler-König and Pilz, 2015, where we will start with the model by Biegler-König and Pilz, 2015, as it is simpler and more similar to the other three models. In this model after constructing a non-overlapping set of Futures and a seasonality curve $s(t)$, the PFC denoted by $S(t)$ is constructed as:

$$S(t) = s(t)\frac{\sum_{u=T_s^i}^{T_e^i} f(u)}{\sum_{u=T_s^i}^{T_e^i} s(u)}$$

where $f(u)$ is the price of the Futures contract covering $[T_s^i, T_e^i]$. This means that if the average price for that period as calculated from the seasonality curve $s(t)$ is equal to 1, and the corresponding Futures price for that period is equal to 2, then the seasonality curve multiplied with 2 gives us the corresponding PFC.

Hagan and West, 2006 look at several methods for the construction of the adjustment function, not only the method used in Caldana, Fusai, and Roncoroni, 2017. They propose a set of qualities they want their curve to have, and then they see what qualities the different proposed curves have. They want to construct yield curves, so their criteria might differ slightly from what one wants when working with electricity prices. Also, the set of available Futures might be different and more complex for yield curves than for electricity leading to greater computational difficulties. We will first state their list of criteria and state how relevant these criteria are for PFCs for electricity prices. We will also see how the other proposed models stand against these criteria.

In West, 2009 a brief summary of the paper by Hagan and West, 2006 is made, listing their criteria and their conclusion. We here state these criteria and afterwards comment on whether or not equivalent criteria are suitable for the adjustment function of a PFC.
(a) In the case of yield curves, is the curve arbitrage free? Thus, we want positivity of the forwards.
(b) In the case of yield curves, how good do the forward rates look? These are usually taken to be the 1m or 3m forward rates, but these are virtually the same as the instantaneous rates. We want as much as possible continuity of the forwards.
(c) How local is the interpolation method? If an input is changed, does the interpolation function only change nearby, with zero or minor spill-over elsewhere, or can the changes elsewhere be material?
(d) Are the forwards not only continuous, but also stable? We can quantify the degree of stability by looking for the maximum basis point change in the forward curve given some basis point change (up or down) in one of the inputs. Many of the simpler methods can have this quantity determined exactly, for others we can only derive estimates.
(e) How local are hedges? Suppose we deal an interest rate derivative of a particular tenor. We assign a set of admissible hedging instruments, for example, in the case of a swap curve, we might (even should) decree that the admissible hedging instruments are exactly those instruments that were used to bootstrap the yield curve. Does most of the delta risk get assigned to the hedging instruments that have maturities close to the given tenors, or does a material amount leak into other regions of the curve?

Equivalent criteria for PFCs for electricity are:

a) Arbitrage free mean in this case that for a positive Futures price, the corresponding day prices are positive, this is as we observe in Figure 3.1 at page 37 not necessarily the case for the method proposed in Benth, Koekkebakker, and Ollmar, 2007. This is because with the number of Futures products used in the construction of the curve, the March Futures product has negative effect on prices in the end of March, meaning a high Futures price in March can lead to negative prices in March. For the other methods one might get negative prices if one Futures product is highly priced in comparison to the other products, as we have a negative spill-over effect for certain days.
b) For modeling interest rates continuity of the curve might be more important than for electricity prices, as we might expect large price differences in short time intervals for electricity. Nevertheless, it seems reasonable that this should rather be exceptions, and that the adjustment curve in general should be continuous also for PFCs for electricity prices.
c) As discussed earlier, this says how great the effect of a change in one Futures price has on the other periods, which is what we show in the special case for the March Future in 3.1.
d) They list how stable their curves are as a criterion, we will later quantify our own measure for the stability of the proposed methods, in their paper they use the measures:

$$||M(r)|| = \sup_t \max_i \left| \frac{\delta r(t)}{\delta r_i} \right|$$

$$||M(f)|| = \sup_t \max_i \left| \frac{\delta r(t)}{\delta f_i^d} \right|$$

which is the maximum of the derivative of the curve with respect to the input, which for us is the Futures prices. As they state, this can not be determined analytically for all methods they study, but we will see that such measures can be determined analytically for the methods we study.
d) This has a relation to point c), for electricity prices, the equivalent criteria will be: If one wants to buy electricity for a day in March, which other Futures products other than the March product will be relevant to hedge this price. This can again be determined analytically for the methods we work with.

3.3.2 Similarities of Models

The ideas behind how to fit the three models to the Futures products are quite different, but as we will show, there are similarities in them as well. Fleten and Lemming, 2003 want to shift the seasonality curve as little as possible while also smoothing it. Benth, Koekkebakker, and Ollmar, 2007 wants to model the difference between the seasonality curve and the PFC with a polynomial spline, and they want this spline to have maximum smoothness based on some smoothness measure. In our Novel model, we argue that if the seasonality curve can represent historical prices, it should also be able to replicate the observed Futures prices. We therefore fit a seasonality curve to the historical prices with the secondary condition that the seasonality curve should directly replicate the observed Futures prices. We will in the following describe is how the seasonality curve is taken into account, and how the PFC changes when the Futures change in the proposed models. We will start by looking at how the PFC changes when the Futures change.

The three methods used to fit the PFC to the Futures products in this study are all similar in the sense they are based on minimizing a squared difference with an added equality constraint. Because of this, we can in all three models express the PFC as a sum of a term dependent on the seasonality curve and a linear combination of the Futures prices, like this:

$$PFC(i) = \tilde{s}(i) + \sum_{i=1}^{n} d_{j,j} F_j(T_j^s, T_j^e)$$

Where $\tilde{s}(i)$ is a normalized seasonality curve, meaning that it averages to zero over each period where we observe a Futures product. The Futures products are denoted by $F_j(T_j^s, T_j^e)$ with $[T_j^s, T_j^e)$ is the period Futures product j covers, and $d_{i,j}$ is the linear factor which describes the sensitivity of Futures product j on electricity with delivery for day i.

To see that our model can be described in such a way, we have to study how the models are constructed. In the three methods discussed earlier, the optimal coefficients $\hat{x}$ are found by a constrained least squares optimization, and can be expressed in this way:

$$\begin{bmatrix} \hat{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^T \cdot A & C^T \\ C & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} A^T \cdot y \\ V \end{bmatrix} \tag{3.1}$$

where V are the Futures prices, y are the historical spot prices, C is the constraints matrix making sure the constraints hold .The vector $A\hat{x}$ will give us the output of our PFC. From A we observe the matrices for the two other methods are similarly defined. Since our models are linear functions with respect to the parameters, and the parameters are linear with respect to the Futures prices, our PFC is linear with respect to the Futures prices. To identify the sensitivity parameter $a_i(t)$ one can simple extract the subset of the right row in the matrix in (3.1), and multiply this with A. We then see we can decompose our coefficients $\hat{x}$ into two parts:

$$\hat{x} = \hat{x}_H + \hat{x}_F.$$

$\hat{x}_H$ is the parameters one would obtain if performing a normal ordinary least square, while $\hat{x}_F$ makes sure the seasonality curve fits to the Futures products. If the Futures prices change, only $\hat{x}_F$ will be affected.

In the method described in Biegler-König and Pilz, 2015, we get that the price for a time period t is only dependent on the Futures product covering that period. In this method the sensitivity parameter $a_i(t)$ is dependent on the seasonality curve. This means there is no spillover effect in this model, this is also the only model that takes the seasonality curve into consideration for the adjustment function. With this model, days when the prices are historically high will vary more when the Futures prices change. It is similar to the Fleten method when $\lambda = 0$, but here all prices are identically shifted, as we here have no spillover effect as well, but this method is also independent of the seasonality curve.

The method described in Hagan and West, 2006 is a Monotone Convex Interpolator, for more details on this and how it compares to the previously mentioned criteria said paper offers a great overview, they also compare it with other models used in the literature for yield curves. The Monotone Convex Interpolator is a second order polynomial spline, where the basis functions are of the form:

$$g_i(x) = a_i x^2 + b_i x + c_i,$$

but where the parameters a_i, b_i, c_1 are chosen dependently on how the different Futures product compare to each other in such a way that the resulting curve is monotonically convex. This means a PFC constructed from this method might be linear when the Futures prices are in a certain interval, but might change behavior at a certain point, resulting in a piece-wise linear relationship between the PFC and the Futures prices. We previously mentioned the other methods could give negative prices as a consequence of the spillover-effect, this can be avoided by using such a method, as we dampen or completely remove the spillover effect if certain Futures prices crosses a certain barrier. For a study on how it used especially for PFCs for electricity prices, see Caldana, Fusai, and Roncoroni, 2017.

We have previously seen that the optimal solution to our two other models are given by a similar matrix. From this, one sees that the optimal coefficients are a linear function of the price of the Future products, denoted by V, and as the PFC is linear in is coefficients, our PFC will be a linear function of the Futures products. In our Novel approach we can separate the coefficients into:

$$\hat{x} = \hat{x}_S + \hat{x}_F$$

where $\hat{x}_S$ is coming from the historical data, and is the coefficients we get when doing an unconstrained least squares optimization. And $\hat{x}_F$ corresponds to the adjustment function, and makes sure our curve is arbitrage free.

As the coefficients are a linear function of the Futures, and our PFCs are linear in the coefficients, the daily price will also be a linear function of the Futures products. This gives us our wanted formula:

$$PFC(i) = a_i + \sum_{j=1}^{n} d_{ij} F_j(T_j^s, T_j^e). \tag{3.2}$$

In the following we will assume our set of Futures products are non-overlapping, and we will simplify the notation by writing

$$F_j = F_j(T_j^s, T_j^e).$$

From 3.2 we get:

$$\frac{\partial PFC(i)}{\partial F_j} = d_{ij},$$

and

$$\frac{\partial^2 PFC(i)}{\partial F_j \partial F_k} = 0.$$

This means that the price of electricity with delivery at day i changes with d_{ij} Euro when the price of Futures product j changes with 1 Euro, and this is independent of the price level of all Futures product. This number d_{ij} is different for the different models. The number $d_{i,j}$ might also be dependent on the number of Futures products we use in the calibration, dependent on which model we use.

How the seasonality curve is constructed in our different models differs slightly. In the Novel method it is assumed that the seasonality curve is estimated simultaneously with the adjustment function, resulting in one curve. The two other methods assume an exogenous given seasonality curve, and thereafter the curve is fitted to the observed Futures prices. Benth, Koekkebakker, and Ollmar, 2007 assumes a clearly distinct seasonality curve and adjustment function by modeling the difference between the seasonality curve and the resulting PFC by a polynomial spline. Fleten and Lemming, 2003 takes the forecasted seasonality curve, and adjusts these prices to the level of the Futures prices while also smoothing them. The three methods can be decomposed like this:

Fleten Method: One starts with a seasonality curve $s(t)$, which is then simultaneously smoothed and fitted to the Futures products in the way described earlier. One can decompose this method by first smoothing the curve, by applying the Hodrick–Prescott filter this method is based on, and afterwards apply the fitting to the Futures.

$$s(t) \to s_{smooth}(t) \to f_{Fleten(t)}$$

Benth Method: In the Benth method one assumes the PFC $f_{Benth}(t)$ can be decomposed in one seasonality curve $s(t)$, and one adjustment curve $\varepsilon(t)$ directly.

$$f_{Benth}(t) = s(t) + \varepsilon(t)$$

Novel Method: In the Novel method we do a simultaneous fitting of the seasonality curve and the adjustment function. The assumption here is that if the mathematical framework used to fitting the seasonality curve is reasonable, it should also be able to reproduce the Futures products. When doing this fitting, we are left with one set of parameters $\hat{x}$, but as seen, these parameters can be split up into:

$$\hat{x} = \hat{x}_H + \hat{x}_F$$

Then our PFC $f_{Novel}(t)$ can be represented as:

$$f_{Novel}(t) = \hat{x}g(t) = \hat{x}_H g(t) + \hat{x}_F g(t),$$

where $g(t)$ is the basis function used for the seasonality curve. In our framework we have chosen $g(t)$ as a trigonometric spline, but any function that is linear in its parameters can be used. This framework is similar to the framework in Benth, Koekkebakker, and Ollmar, 2007, but they allow for different functions used for the adjustment function and the seasonality curve.

3.3.3 Differences between the models

In the previous sections we have discussed what is similar between the different construction methods used for the PFC. We shall now describe the differences between these models. The first difference to point out is that in the model proposed by Benth, Koekkebakker, and Ollmar, 2007, the number of parameters depends on the number of Futures products observed. As shown earlier, this leads to a deterministic change in the curve when a new Futures product is added to the market. As we know beforehand that this product will be traded, and we already have an estimated price for this product from our PFC, we can use this information to arbitrage the curve, which we will show later.

A possible solution to this would be to extend our spline to the maximum number of knots needed from the beginning. Instead of starting with a set of knots corresponding to the currently observed set of quarterly and monthly Futures products, one could use a set of knots that lets us replicate all potential Futures products. One will then end up with a smoother curve, with respect to the relevant measure[1], as such a curve will be at least as smooth as the original curve. If this curve replicates a Futures product that is not yet traded, but will be in the future, then this curve will not have an incentive to change with the inclusion of this product in the optimization. The downside is off-course that this leads to a larger set of parameters, which in turn leads to a linear problem in higher dimensions, especially when working with curves covering several years. As the solution to the original problem is typically found with numerical techniques, this will also be the case for the extended case, and this might lead to computational difficulties and only approximate solutions.

[1] $\min_x \int_{t_0}^{t_n} [\varepsilon''(t,x)dt]^2 dt$

The smoothing in the different methods is also taken into account. Fleten and Lemming, 2003 smooths the seasonality curve, basically assuming the seasonality curve is not smooth enough. We have earlier shown this helps smoothing out the gap between months when working with dummy variables, but it can also suppress the daily/hourly seasonality. We have earlier shown how we can decompose the smoothing of the seasonality curve and the adjustment function. This means we can either chose to only smooth the seasonality function, and choose some other adjustment function, or only chose adjust the curve to the observed Futures. One could also choose different λ values for the two different parts, giving more flexibility. Benth, Koekkebakker, and Ollmar, 2007 assumes the original seasonality curve is smooth enough and only ensures that the adjustment function itself is smooth. The Novel method does not consider smoothness as a factor in the optimization, apart from the fact we want our curve to be continuous with continuous derivatives.

In the novel method we propose a combined fitting of the seasonality curve which directly fits the curve to the level of the Futures prices. In the current version we have taken monthly Futures prices as the smallest Futures product into account, if we want to construct a curve that is consistent with products with shorter granularity, we will need to include more parameters to the model. With this approach more parameters leads to more free parameters which can be used for the fitting of the seasonality curve. Since we do not always observe all Futures products, we are free to use these parameters to estimate the seasonality function, leading to a higher probability of overfitting in this approach. The possibility to fit our curve to all available Futures products at all times seems appropriate, as we remove the arbitrage opportunity when new products are introduced to the market. Nevertheless, we do not want to keep all those extra free variables available for fitting the seasonality curve. Earlier we observed that the increased amount of parameters in our Novel method might lead to overfitting as when testing our models against realized spot prices the Novel method was superior for in-sample testing while the Fleten method was the best for out-of-sample testing.

3.4 Dynamics of the PFC

The PFC as a curve needs to change in time as we come closer to maturity, as the price of the traded Futures products change, and the number of observed objects change. In Biegler-König and Pilz, 2015 they argue for why one should only update the seasonality curve, in their paper called shape, on an infrequent but still regular basis. They base this on the fact that the updating of the shape can be numerically expensive. That historical data will only significantly affect the shape if the added time interval is of a certain size. Altering the shape will lead to profit and loss jumps when pricing or risk managing products or portfolios. They do not specifically state how often the seasonality curve is updated, so we will assume this is remained constant for the whole length of our study.

In the following sections we will handle the PFCs as purely financial objects, and we will assume full information of it at all times. We will assume all days can be traded liquidly from the PFC, which also means one can implicitly buy Futures product that are not in the market yet, by buying the corresponding days from the PFC. These Futures products we will call implicit Futures products, as they are implicitly given from the PFC. It is clear that when a Futures product is traded in the market, the price of this is equal to the price of the implicit Futures product. These assumptions are not realistic for a trader in the market, as he will get a price for a specific load curve, and this price is dependent on a bid-ask spread for the Futures prices. Nevertheless, a producer of a PFC will have this information when he constructs his PFC, and any weaknesses shown here, the producer would himself strive to avoid.

We will talk about forward prices and Futures prices in this study, where we define the forward price as the price generated by the PFC, and is typically a price for one day in the Future. The Futures price is the price of a Futures product that is observed and traded today at the EEX, this is covering a certain time-period. We will restrict our study by only taking monthly, quarterly and yearly products into account, and only considering products for the year of 2015. In reality one would take more products into account, especially for the short term of the curve, and one would also construct a curve for several years. We do this to simplify our framework, but we also want to study the difference between a PFC constructed for one year, or the same year as taken from a PFC constructed for multiple years. By only taking this subset of products into account we can still observe the most important features of the dynamics, which is what happens when the price change, and what happens when a product is traded. What we wont observe is what happens when a new yearly contract is added in the long-end of the curve, or a product reaches maturity, and is therefore not traded. From our results we will see how this can and should be taken into account as well.

3.4.1 Notation

In our framework we go from looking at the PFC as a static object where the number of Futures and the value of the Futures are held constant, to a dynamic object where the number of traded Futures products and the value of these products are changing. This change in the framework needs a new and consistent terminology. We will first go through the relevant terminology used:

Earlier we denoted the PFC by $PFC(\cdot)$, where:

$$PFC : [1, 365] \to \mathbb{R} \tag{3.3}$$

meaning that PFC takes a day in a year, and returns the forward price for that day. As we are now working in a dynamic setting, we have to specify the number of Futures products taken into account, as this changes when time to maturity changes, and the price of these Futures products. As we are now seeing the PFC as a function, dependent on time, we will also use the notation f instead of PFC. Therefore, in our setting we set:

$$f_j : [1, 365] \times \mathbb{R}^n \to \mathbb{R} \tag{3.4}$$

where j denotes which day the PFC is computed at and n is the number of Futures products traded at day j. Therefore, f_j takes as input the day in the future the forward price should be computed for, and the Futures prices observed at day j. We will from now on use the notation:

$$f_j(i, V_j^n) \tag{3.5}$$

When we are not talking about a specific day, we will use the notation:

$$f_j \cdot, V_j^n \tag{3.6}$$

to imply the whole PFC.

j: Day the PFC is estimated at, which decides the number of Futures (n) used. From Table 3.1 on page 60 we see an overview of the number of Futures traded at the different day.
We see that for the days $1-61$ we observe 2 Futures products, meaning for $j \in [1, 61]$, $n = 2$. For days $62 - 123$ the number of Futures products are 3, meaning $n = 3$ when $j \in [62, 123]$, and so on.

i: Day in the future the forward price is calculated for.

Δ_k: Is the subset of days covered by Futures product k in 2015. In our framework this will be a month, quarter or year.

v_k^j: price of Futures product covering period Δ_k on day j.

V_j^n: The set of the observed Futures products observed at day j, defined as:

$$V_j^n = \{v_1^j, \cdots, v_n^j\} \tag{3.7}$$

By the no-arbitrage condition we get this relationship between the Futures prices and the PFC:

$$\frac{1}{|\Delta_k|} \sum_{i \in \Delta_k} f_j(i, V_j^n) = v_k^j \tag{3.8}$$

meaning the average price of the PFC, over the time-period Δ_k relating to a Futures product has the same value as the price v_k^j of that Futures product.

Similarly we can get an implied price for Futures products that are not yet traded, but will be traded in the Future by splitting a period Δ_k into two or more subperiods, $\tilde{\Delta}_k^1$ and $\tilde{\Delta}_k^2$, giving us the two implied Futures prices $\hat{v}_{k,1}^j$ and $\hat{v}_{k,2}^j$ which are defined as:

$$\frac{1}{|\tilde{\Delta}_k^g|} \sum_{i \in \tilde{\Delta}_k^g} f_j(i, V_j^n) = \hat{v}_{k,g}^j : \ g = 1, 2. \tag{3.9}$$

This will give us an implied set of Futures, denoted by $\hat{V}_j^n$, defined by:

$$\hat{V}_j^n = \{(v_1^j, \cdots, \hat{v}_{k,1}^j, \hat{v}_{k,2}^j, \cdots) | \hat{v}_{k,i}^j : \quad \text{implied price of Futures product covering period } \tilde{\Delta}_k^i \text{ on day j}\} \tag{3.10}$$

where the implied price is equal to the original price if one does not split a period. One can split the set of Futures into an arbitrary number of products, not only into implied Futures products that will be traded in the future. We will in our study restrict ourselves to splitting quarterly products into monthly products to give a picture of what happens before and after a monthly product is introduced to the market., but in general we will split the Future so it corresponds to how the market will be the next time a new Futures product is added.

We will also use the derivative of the forward price with respect to the Futures prices v_k, which defines how much the forward price increase when the Futures price increase, this will be denoted as:

$$\frac{\partial f_j(i, V_j^n)}{\partial v_k} = d_{i,j}^k(V_j^n) \tag{3.11}$$

3.4.2 Derivative of the PFC

As we stated in the previous section there is a linear relationship between the PFC and the Futures products, and this is true for all three methods considered in this chapter. Since there is a linear relationship, one can easily find the derivative of the PFC with respect to the Futures, and it is given by:

$$\frac{\partial f_j(i, V_j^n)}{\partial v_k} = d_{j,i}^k \tag{3.12}$$

$$\frac{\partial^2 f_j(i, V_j^n)}{\partial v_{k_1} \partial v_{k_2}} = 0 \tag{3.13}$$

where j is the day we calculate the PFC from, which decides the number of Futures we observe, i is the day we calculate the derivative for and n is specifying the Futures product we derive with respect to.

As a result of this, we get this connection between the price of the PFC at different days and the change in the price of the Futures:

$$f_{j+1}(i, V_{j+1}^n) = f_j(i, V_j^n) + \sum_{k=1}^{n} d_{j,i}^k v_{\Delta k}^j$$

if one assumes that the number of Futures products traded are the same at both days. We define $\Delta v_k^j = v_k^{j+1} - v_k^j$ as the change in the price of Futures product k between day j and day $j+1$. As one can implicitly buy Futures products that are not yet traded at the EEX, it would be natural that a similar relationship holds, even if the granularity of the Futures products change.

3.4.3 Hedging of the PFC

As we have shown, when the number of Futures are constant, the price of electricity tomorrow as a function of todays price and the change in the Futures prices are given by the formula:

$$f_{j+1}(i, V_{j+1}^n) = f_j(i, V_j^n) + \sum_{k=1}^{n} d_{j,i}^k \Delta v_k^j$$

By rearranging the terms, one sees that:

$$C = f_j(i, V_{j+1}^n) - \sum_{k=1}^{n} d_{j,i}^k v_k^{j+1} = f_j(i, V_j^n) - \sum_{k=1}^{n} d_{j,i}^k v_k^j$$

This means, for a trader only looking to trade in the PFC for financial reasons who assumes the PFC is constructed by such a linear model will be indifferent to trade in the PFC or a corresponding set of Futures prices. This relationship will hold so long the number of observed Futures products is constant. The reason for this is that he can chose instead of investing an amount in the PFC, he can spend the same amount in a corresponding portfolio of Futures products, and these Futures products will because of the linear relationship between the Futures and the PFC always perfectly hedge the PFC.

This is not the case if there is no such linear relationship between the PFC and the Futures, as then one needs a different amount of Futures products when the price changes to hedge the PFC. In Hagan and West, 2006 they investigate how several non-linear curves can be hedged.

3.4.4 Arbitrage opportunities

As we have shown, one can hedge the PFC by buying a suitable portfolio of Futures products so long the number of Futures is constant. This is not the case if one buys a electricity from the PFC curve at time $t_1 < T$ and sells at time $t_2 > T$, where T is some point in time when a new Futures product is introduced to the market. This is clear since the products needed to hedge the curve at time t_2 are not necessarily traded at time t_1, and one can not hedge against this risk by trading in the Futures products available at time t_1. What we want to show in this section, is the arbitrage opportunity that appears when working with the Benth method, when allowing to implicitly trade in the product introduced at time T, by buying implicit Futures products $\hat{v}^k_{\tilde{\Delta}^i_j}$ from the PFC.

The strategy is as follows:

Step 1: On day t_1 we observe n Futures products $V^n_{t_1}$ and from this we construct a PFC $f_{t_1}(\cdot, V^n_{t_1})$ by using the method proposed by Benth. From this PFC we observe $n+1$ implied Futures products $\hat{V}^{n+1}_{t_1}$, obtained by splitting the fourth Futures product corresponding to the second quarter into a one-month and a two-month Futures product, which is the set of products we will observe at day T and t_2.

Step 2 : From the implied set of Futures products we re-estimate the adjustment function of the PFC with the Benth method, keeping in mind we need more parameters to account for the new implied Futures product. We then obtain the PFC $f_{t_2}(\cdot, \hat{V}^{n+1}_{t_2})$. Both PFCs will be arbitrage to both the original set of Futures products, and the new implied set.

Step 3: In the two PFCs there are periods where the prices do not coincide, as observed in 2.5. Chose a day i where the inequality

$$f_{t_2}(i, \hat{V}^{n+1}_{t_2}) > f_{t_1}(i, V^n_{t_1}) \tag{3.14}$$

holds. If the Futures prices to not change between time t_1 and time T, then just buying the product $f_{t_1}(i, V^n_{t_1})$ will result in a profit. The Futures prices will almost surely vary between t_1 and T and we will then need to hedge this risk by buying a corresponding portfolio of Futures products.

Step 4: To hedge for the fact that the price of the Futures prices might change, we need to trade in the set of implied Futures. The idea is that at time t_2 there exists a portfolio of Futures which replicate the price $f_{t_2}(i, V_{t_2}^{n+1})$, given by:

$$f_{t_2}(i, V_{t_2}^{n+1}) = a_i^{t_2} + \sum_{k=1}^{n+1} d_{i,t_2}^k v_k^{t_2}$$

this means one has $a_i^{t_2}$ in the bank account, and holds d_{i,t_2}^k numbers of Futures product k with the price $v_k^{t_2}$. If one shorts the same portfolio of Futures product at time t_1 from the implied Futures products given by the PFC $f_{t_1}(\cdot, V_{t_1}^n)$ the total investment at time t_1 will be:

$$I_{t_1} = f_{t_1}(i, V_{t_1}^n) - \sum_{k=1}^{n+1} d_{i,t_2}^k \hat{v}_k^{t_1} \tag{3.15}$$

The value, and potential sales price, of this investment at time t_2 will then be:

$$V_{t_2} = f_{t_2}(i, V_{t_2}^{n+1}) - \sum_{k=1}^{n+1} d_{i,t_2}^k v_k^{t_2} \tag{3.16}$$

Assuming the implied Futures are one-to-one tradeable with the Futures at the EEX, the values of the Futures products will cancel out and the profit will be:

$$V_{t_2} - I_{t_1} = f_{t_2}(i, V_{t_2}^{n+1}) - f_{t_1}(i, V_{t_1}^n) > 0$$

This example is not a realistic example that could be used to make an arbitrage strategy for several reasons. First of all this strategy assumes full knowledge of the PFC, and the construction of this from all participants, while in reality only a buyer of electricity will only receive the prices for a set load curve. Secondly the PFC is usually used to trade physical electricity, and not for financial settlement, as we have assumed here. Thirdly, in a real world setting, the retailer will probably work with different prices for buying and selling electricity corresponding to the bid-ask spread of Futures products, which will remove the possibility of such arbitrage opportunities.

This does however show that a producer of an electricity curve should be cautious of adjusting the PFC to the level of the Futures prices by a method where the number of parameters are dependent on the number of Futures products observed. In the paper by Hagan and West, 2006 they propose several such methods where the number of parameters are dependent on the number of products observed. One could find similar shortcomings in these models, but they are harder to take advantage of, as these models are not necessarily linear as a function of the Futures products. Therefore, the resulting arbitrage opportunity is in general not hedge-able in the same way as for a model which is linear in the parameters.

Numerical Example

We will here show the previously mentioned arbitrage opportunity from the method used in Benth, Koekkebakker, and Ollmar, 2007. We will work with the residual Futures prices, which are the Futures prices minus the average of the seasonality curve for the corresponding period. We will follow the steps outlined earlier, with real observed data. On day j, which is a certain trading day in 2014 we observe 6 Futures prices for 2015, while at day $j+1$ we observe 7.

Step 1: At day j we observe the Futures prices

$$(38.22,\ 39.70000,\ 34.07,\ 31.85\ 33.47,\ 36.69),$$

denoted in euro/MWh. And we denote the residual Futures prices as V_j^6, where

$$V_j^6 = (-11.45,\ -0.65,\ -15.60,\ -11.52,\ -14.14,\ -10.92).$$

From this we construct our PFC $f_j(\cdot, V_j^6)$ and we construct the 7 implied Futures products $\hat{V}_j^7$, obtained by splitting the second quarterly product into a one and a two-month product. This implied Futures product is

$$\hat{V}_j^7 = (-11.45,\ -0.65,\ -15.60,\ -18.957,\ -7.86,\ -14.14,\ -10.92).$$

Note that $(-18.957 \cdot 30 - 7.86 \cdot 61)/91 = -11.52$, and all other Futures products are equal.

Step 2: From the implied set of Futures, we construct the implied PFC $f_{j+1}(\cdot, \hat{V}_j^7)$ [2]. Both curves $f_{j+1}(\cdot, \hat{V}_j^7)$ and $f_j(\cdot, V_j^6)$ are then arbitrage free to both sets of Futures products.

Step 3: Find a day i where

$$f_{j+1}(i, \hat{V}_j^7) > f_j(i, V_j^6).$$

In Figure 3.3 we show the two adjustment functions, and see there exists such a day. We observe that on day 27 this difference is the biggest, the difference is then 5.55 euro/MWh. If the Futures prices do not change, but only cascade into the new products, buying 1 MWh of electricity for day 27 will result in a profit of 5.5 euro.

Step 4: At day $j+1$, we observe the Futures prices

$$(38.07,\ 39.25,\ 34.10,\ 32.57,\ 31.64,\ 33.33,\ 36.55),$$

and we denote by V_{j+1}^7 the residual Futures prices

$$V_{j+1}^7 = (-11.60,\ -1.10,\ -15.57,\ -16.87, 16.63, 14.28, 11.06).$$

As $V_{j+1}^7 \neq \hat{V}_j^7$, we need to hedge for the fact that the Futures prices are changing. At day j we make the investment

$$I_j = f_j(i, V_j^6) - \sum_{k=1}^{7} d_{i,j+1}^k \hat{v}_k^{j+1}.$$

Which means one buys from the PFC electricity for day i, and shorts the corresponding portfolio of implied Futures products[3]. At day $j+1$, the value of this investment

[2]We need the subscript $j+1$ on our functions f, as we now use 7 Futures products as input

[3] This is only possible if you have full information of the curve, and you are allowed to trade from the implied PFC.

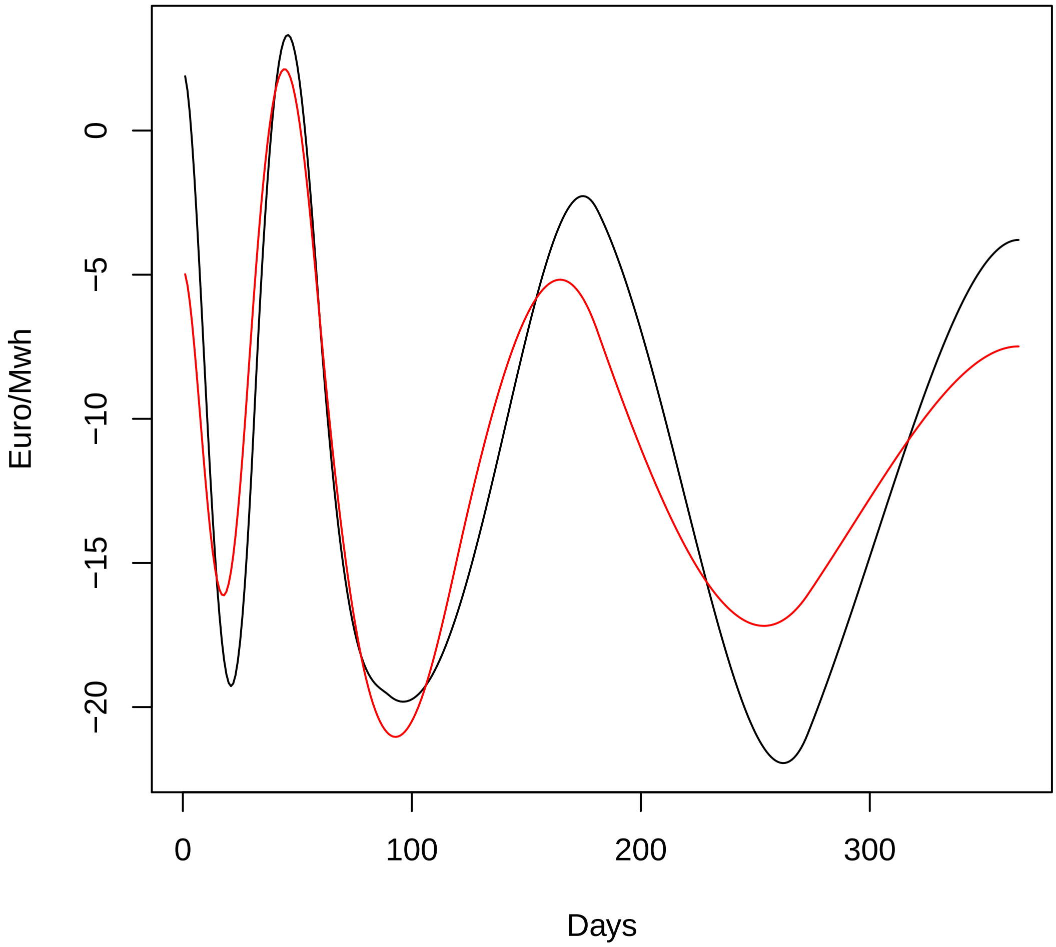

FIGURE 3.3: Showing the respective adjustment curves with the original 6 (black), and 7 implied (red) Futures

is

$$V_{j+1} = f_{j+1}(i, V_{j+1}^7) - \sum_{k=1}^{7} d_{i,j+1}^k v_k^{j+1}.$$

We know that

$$f_{j+1}(i, V_{j+1}^7) = f_{j+1}(i, \hat{V}_j^7) + \sum_{k=1}^{7} d_{i,j+1}^k (v_k^{j+1} - \hat{v}_k^{j+1}),$$

as it is linear in the Futures products. Therefore, we get

$$V_{j+1} - I_j = f_{j+1}(i, \hat{V}_j^7) - f_j(i, V_j^6) = 5, 5.$$

The total profit is then 5.5 euro/MWh, and the risk of changing Futures prices is hedged.

$$-11.45,\ -0.65,\ -15.60,\ -18.957,\ -7.86,\ -14.14,\ -10.92$$

From this we learn that while constructing the PFC, one should rather take into account the maximum amount of information one will use to construct the PFC, and not the information available at the moment. So for the case of Futures, one should see how many Futures products will be available at one point in the future, and not what information is available now, when making the adjustment function.

3.4.5 Spillover Effect

We have earlier mentioned what we call the spillover effect, which is the effect a certain Futures product has on prices outside this period. In this section we will go into detail on this effect for the mentioned methods, discussing the natural conditions for the spillover effect.

Definition 3.4.1. *For a price forward curve $f_j(\cdot, V_j^n)$, the spillover effect $S_j^k(T_i^s, T_i^e)$ is defined as the effect a change in Futures product k has on the price of electricity for the period $[T_i^s, T_i^e)$. If $[T_i^s, T_i^e)$ corresponds to a traded product v_g^j; $g = 1 : n$, then the spillover offect is 1 if $g = k$, for other values of g the spillover effect is 0. For any period $[T_i^s, T_i^e)$ not corresponding to a traded product $S_j^k(T_i^s, T_i^e) \in \mathbb{R}$*

It is worth noticing that this is the direct effect a change in one Futures price has on individual days in periods not covered by this product. A natural assumption is in many cases that there is some correlation between the Futures products, as the underlying price factors for certain periods are the same. Meaning a decrease for prices in June often means the prices during July also fall and vice versa. This should not be taken care of in the adjustment function, as this shows the direct effect an increase of a certain Futures product has on the individual prices, but rather be accounted for in a correlation structure for the statistical model of the Futures prices. When working with models which are non-linear, this analysis is of course harder to quantify, as the spillover effect will be dependent on the current level of the different Futures products.

As we see from figure 3.1 at page 37 a change in the price of any Futures product, here illustrated with the March Future, will give an effect on the prices in the other periods. The size and direction of this change is dependent on the model used. We will first mention what characterizes the spillover effect in the three models discussed, and as a special case what happens in the model discussed in Biegler-König and Pilz, 2015, as the spillover effect here is easily characterized. From this infer what seems natural conditions for the spillover effect. Hagan and West, 2006 discuss something similar as the spillover effect for their models, we will also include their conclusion as a comparison. We will use figures 3.1, 3.9 and 3.10 to illustrate our points. The first figure shows how the prices change with respect to a small change in the March Futures prices. The second figure shows how this change is dependent on the number of other Futures products we observe. The last figure shows how the spillover in the Fleten model is dependent on the λ-value.

Fleten method: In the Fleten method we observe the least amount of spillover effect, and the amount of spillover effect we observe is getting close to 0 after 30 days. So a change in the March Futures price will mostly affect prices in February and April. From Figure 3.10 at page 68 we observe that the higher the λ-value, the more spillover effect we observe. We also know if $\lambda = 0$, the spillover effect is equal to 0, and all prices in the corresponding period will be changed with the same amount. Also, in this method the spillover effects seems to not be heavily affected by the amount of Futures products observed. This will be different if we include artificial Futures products only covering the beginning of April or end of February.

Benth Method: In the Benth method we observe the largest spillover effect, especially for dates before March. We also see from Figure 3.9 67 that the number of Futures products determines the amount of spillover effect and how it distributes itself. We see that the more products we observe the spillover effect is decreasing since the periods where the spillover effect needs to average to 0 over gets smaller. We would also observe different spillover effects if we include more yearly products in the long end of the curve.

Novel Method: In the novel method the spillover effect is also present for all periods, but seems smaller than for the Benth method. In this method the spillover effect is not dependent on the number of Futures observed, but this is only because we from the start take into account what kind of Futures products we will observe, because of this we can't use this method for an arbitrary amount of Futures objects, as it has a parameter restriction.

Shifting algorithm: We include the shifting algorithm described in Biegler-König and Pilz, 2015 as it is the only method with no spillover effects. In this method they only shift the prices directly affected by the change in the Futures price by a multiple of the seasonality curve, as described earlier. And keeps the prices in all other periods constants. Because of this, the price of any day in March will only change when the March Futures price change etc.

Monotone Convex Interpolator: In the paper by Hagan and West, 2006 they examine several methods, where not all of them are necessarily linear in the Futures products, among them the Monotone Convex Interpolator (MCI) recently discussed in Caldana, Fusai, and Roncoroni, 2017. They also discuss how local these methods are, and how local the hedges are, which are two sides of the same coin. They conclude the MCI method has little spillover effect and is locally hedgeable, but they do not give an exact representation of what this means, as we get when our models are linear as a function of the Futures products.

Considering this, what are natural conditions for the spillover effect, and how can we model it. First of all, there is no rule that says we need a spillover effect, which is the case in the Fleten method when $\lambda = 0$ or the shifting algorithm. The biggest disadvantage from this is that no spillover-effect typically means a non-continuous PFC, as we will typically adjust prices during at the end/start of two consecutive periods differently, implying a jump in prices. This weakness can be overcome by assuming adjustment function continually goes to zero for the end points of the adjustment function, but this implies that these prices are not affected by the uncertainty of the Futures prices, which also seems insufficient. Because of this, we will assume a spillover effect is natural.

3.4.6 Sensitivity of the Adjustment Function

In Figure 3.2 we observed how the price for the 30th of June changes as we come closer to delivery for the Benth, Fleten and the novel model. This change is either because the price of the observed Futures products change, or because the granularity of these products change. What we observe is that the prices seems to evolve in a similar manner for the method discussed in Fleten and Lemming, 2003 and the novel method. In the model discussed in Benth, Koekkebakker, and Ollmar, 2007 we observe similar dynamics, but occasionally rather large jumps at the points where new Futures products are added to the market. This difference between the models is coming from the fact that in Benth, Koekkebakker, and Ollmar, 2007 the number of parameters are changing when the number of Futures products change, leading to a deterministic change in the PFC. It might be natural to assume bigger jumps in the two other models as well when we introduce a new Futures product, as there is some miss-pricing of this product, we will later explain why this is not the case for the Fleten method.

In this section we want to discuss how much the PFC changes, when the different Futures product change. We will discuss this by introducing a sensitivity measure. Our sensitivity measure is defined as follows:

$$D_k^j = \frac{\sum_{i=1}^{365} |d_{i,j}^k|}{\sum_{i=1}^{365} d_{i,j}^k}. \tag{3.17}$$

We recall that $d_{j,i}^k$ is the change in the price of electricity at day i, when Futures product k change, and the number of products observed are denoted by j, which indicates the day we observe the change at. This means the numerator is change of the total change in the PFC, when product k is changing, independent of the direction the curve is changing. The denominator is a normalizing variable dependent on the length of Futures product k. If $D_k^j = 2$, we have as much change outside period k as a result of the spillover effect, as in the period. If $D_k^j = 1$ we have no spillover effect at all.

There are several variables that can affect how the adjustment curve behaves, and this will also affect any sensitivity measure. The number of Futures products will in the method by Benth, Koekkebakker, and Ollmar, 2007 always affect the adjustment function. In the method by Fleten and Lemming, 2003, it will affect the adjustment function if the new product is added before the cut-off point of any already observed product. In the novel method, we will not have a dependency on the number of observed products, as we have pre-specified the number of products we can include in our construction. Also, in the Fleten method the λ-value makes an effect on the spillover-effect, where typically a higher λ-value means higher spillover effect. We will here try to explain how sensitive our models are, and see how the different parameters affects this sensitivity.

In Table 3.2 at page 61 we show how the value of D_k^j differs for the three proposed methods, when assuming we observe 12 monthly Futures. We observes that in the Fleten method one has by far the smallest spillover effect, with an average value lower than 1.10. The two other methods show similar amounts of spillover effect, where the Benth method in sum has a slightly higher value. In the Benth method we observe slightly higher spillover effect in the beginning, which we contribute to

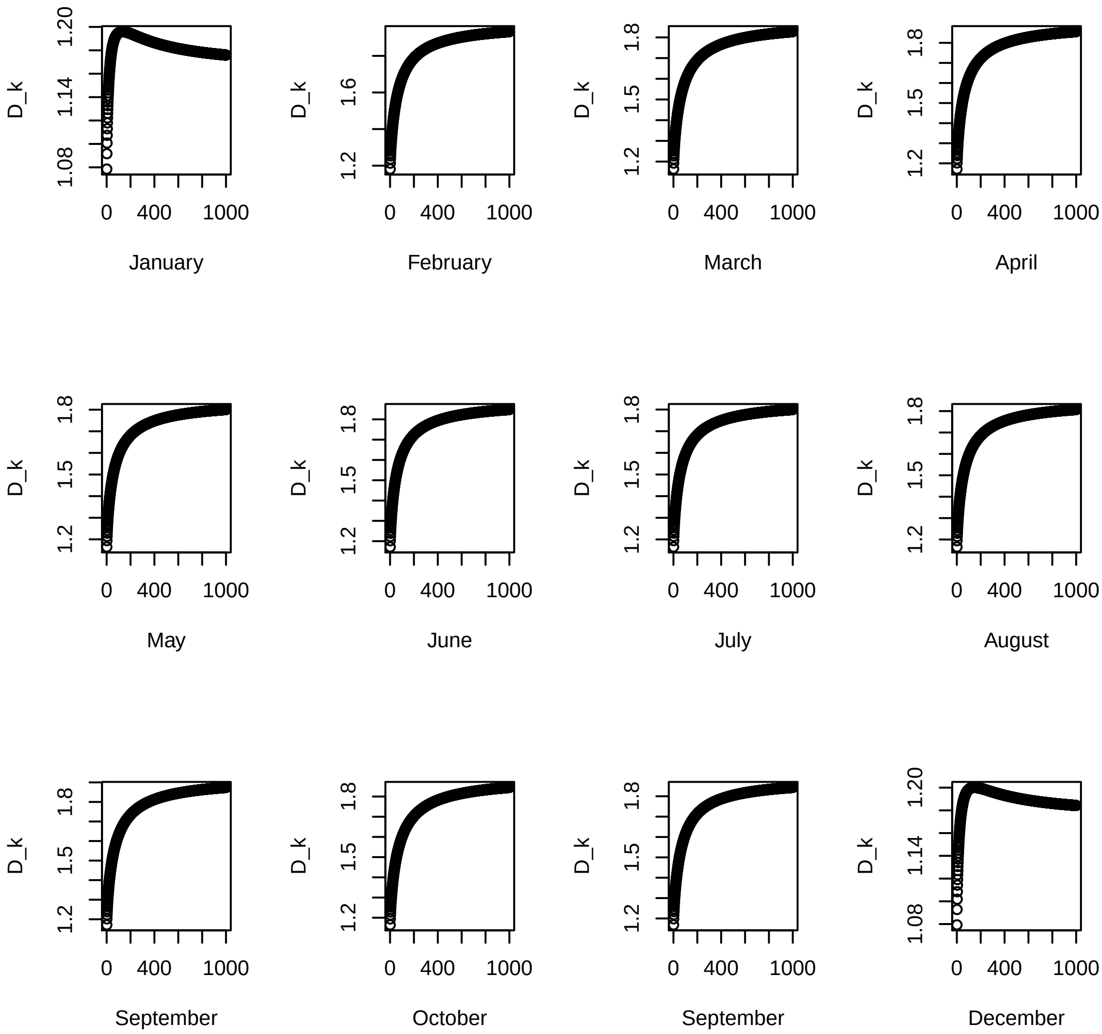

FIGURE 3.4: The sensitivity measure of the Fleten curve with monthly Futures when λ is between 100 and 100.000.

the extra restraint on $\varepsilon'(t_n)$ in the long end of the model proposed in Benth, Koekkebakker, and Ollmar, 2007. The Novel method shows equal amount of spillover effect for all months since this method is symmetric, and has no special constraints on the short or the long term of the curve.

In Plot 3.4 at page 58 we see how the sensitivity measure varies with the λ-parameter in the Fleten method when assuming we observe monthly Futures products and λ-values between 100 and 100.000. We observe that for all periods apart from the end periods we have more sensitivity with respect to the Futures when λ is increasing. For the edge periods, correspondingly January and December, we observe a maximum when $\lambda = 13.400$ and thereafter a decrease. For the other periods we observe an approximately logarithmic increase in the sensitivity, which seems to flatten out at about 2.0, which is still less than the sensitivity in the two other methods. This behavior comes from the fact that we only have continuity constraints on one side,

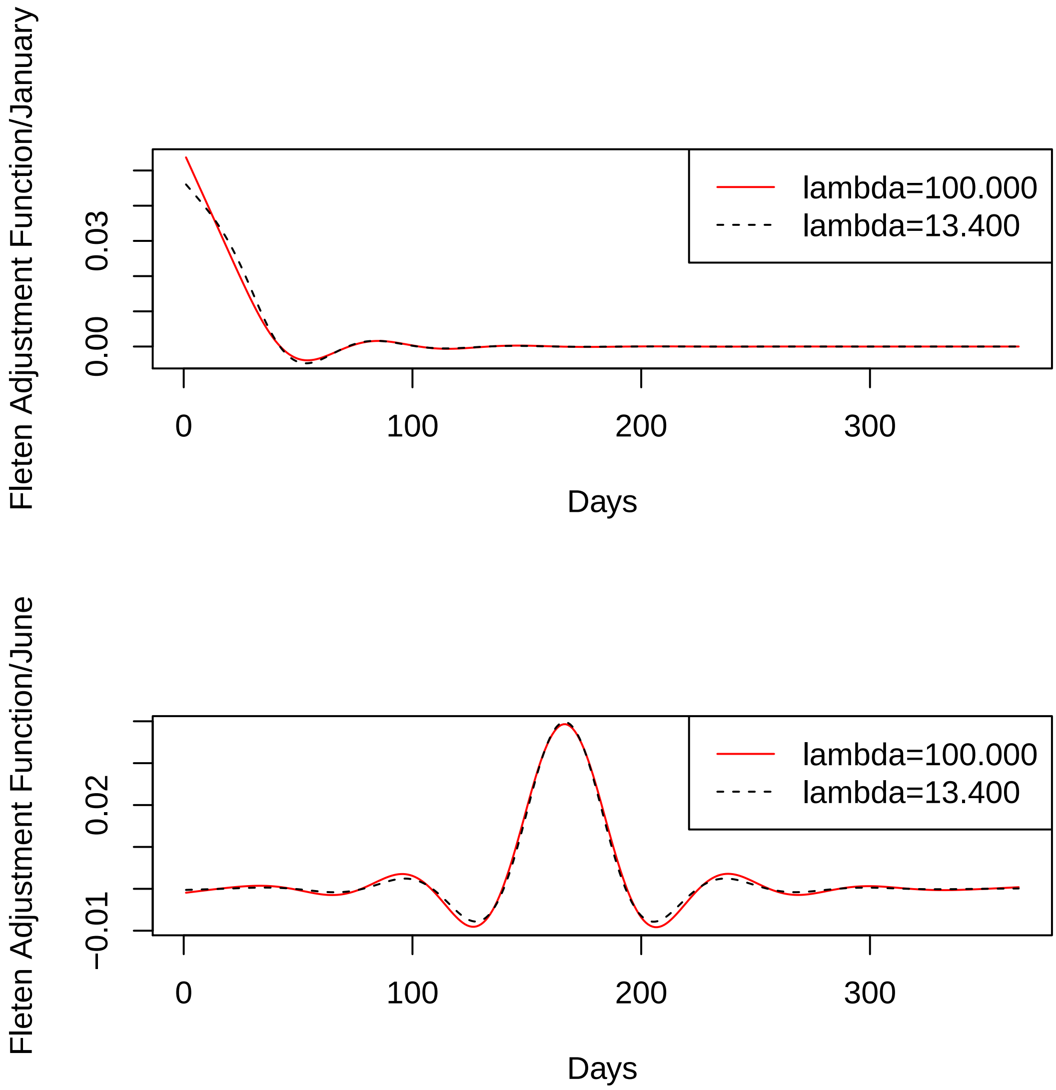

FIGURE 3.5: The Fleten Adjustment function for January and June for $\lambda = 100.00$ and $\lambda = 13.400$ respectively.

so one can have a much steeper decline for these periods. This is shown in figure 3.5, where we observe that for a high λ value we get a steeper decline for the month of January, which is not possible for the month of June, as we have restraints on both sides.

In Figure 3.6 at page 60 we show how the distance between the PFCs constructed at two consecutive days are throughout the year. We use the L2 norm[4] as a measure here. Since we observe large differences in the differences, we use different scales for the different methods. As expected from Figure 3.2 the sensitivity of the Fleten method is the smallest, while the sensitivity of the Benth method is by far the largest. We also observe in all models a jump at times when new Futures products are added. This jump comes from the fact that the original PFCs miss-price the new Futures product, and therefore when this is added to the market, the prices are forced up or down to cope with this miss-pricing. As stated earlier, this just was not

[4] $\sum_{i=1}^{365}(f(i,j+1) - f(i,j))^2$

Days	Knots
1-61	(1-90,91-365)
62-123	(1-90,91-181,182-365)
124-148	(1-31,32-90,91-181,182-273,274-365)
149-189	(1-31,32-59,60-90,91-181,182-273,274-365)
190-213	(1-31,32-59,60-90,91-120,121-181,182-273,274-365)
214-252	(1-31,32-59,60-90,91-120,121-151,152-181,182-273,274-365)

TABLE 3.1: Table showing the set of Futures we take into consideration at the different trading days in 2014. For trading days 1-61 we observe a yearly product and the first quarterly product, while for the days 124-148 we also observe the first monthly product. For these days we split the quarterly product in two because of the no-arbitrage requirement.

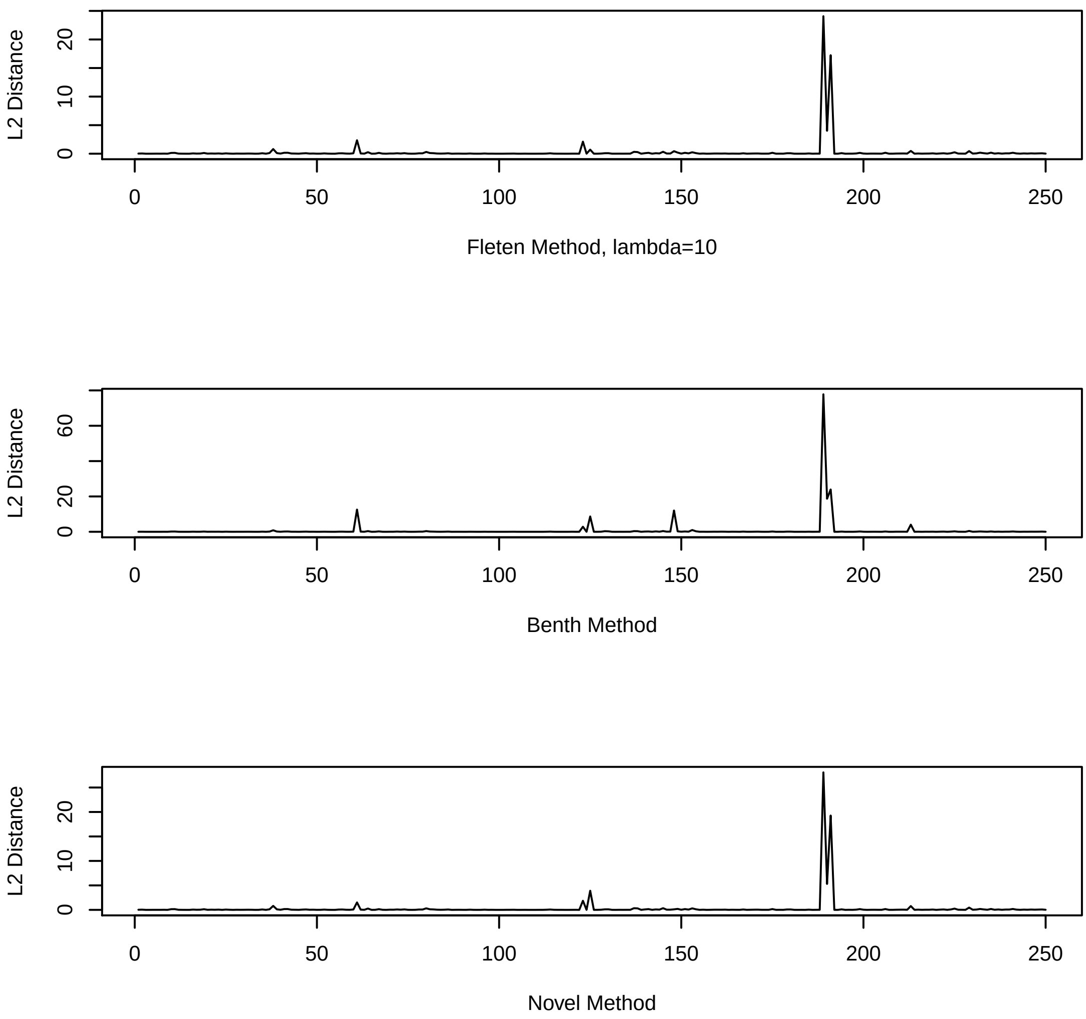

FIGURE 3.6: The distance between corresponding PFCs, as measured by the L2 norm between two following curves.

Month	Fleten method, lambda=10	Benth Method	Novel Method
1	1.04	1.86	**1.98**
2	1.10	**2.72**	2.11
3	1.09	**2.66**	1.98
4	1.09	**2.88**	2.02
5	1.09	**2.58**	1.97
6	1.09	**2.29**	2.04
7	1.09	**2.01**	2.00
8	1.09	1.76	**1.99**
9	1.09	1.63	**2.04**
10	1.09	1.71	**2.00**
11	1.09	1.71	**2.02**
12	1.04	1.23	**1.99**
Sum	13.02	**25.05**	24.13

TABLE 3.2: Table showing the value of D_k^j for the three methods proposed earlier, when taking all 12 monthly Futures as input

observed in the Fleten or novel method when only looking at the 30th of June price, but we observe it when considering the whole curve.

We can explain this as follows: If we work with values for $d_{i,j}^k$ normalized to 1, and we consider the case where we have either 11 or 12 Futures products, where we split the June-July product into two when working with 12 products. The 30th of June corresponds to $i = 181$, and in this method we see from 3.1 at Page 37 that for the edge of a month the derivative is approximately half off the max value. Therefore, both $d_{181,6}^{12}$ and $d_{181,7}^{12}$ is close to 0.5, say 0.55 and 0.45 respectively. If the PFC constructed with 11 products miss-price the June Futures product with an amount k, the July product will be miss-prices with $-k$[5]. When the product covering both June and July is cascading, the PFC needs to correct for the miss-pricing of June and July, the effect on the 30th of June is then

$$k \cdot d_{181,6}^{12} - k \cdot d_{181,7}^{12} = 0.1 \cdot k.$$

Therefore, the 30th of June is only affected by 10% of the miss-pricing of the June/July products. For the middle of June, say $i = 165$, we observe $d_{165,6}^{12} \approx 1$ and $d_{165,7}^{12} \approx 0$, and the miss-pricing of the June Futures price will affect day 165 with

$$k \cdot d_{165,6}^{12} - k \cdot d_{165,7}^{12} \approx 1 \cdot k.$$

These numbers are just approximations, but they are close to the real numbers observed in the model. For the Fleten model, the value $d_{181,6}^{k}$ and $d_{181,7}^{k}$ is not affected by whether we observe monthly products corresponding to $k = 12$, or one product covering the whole of August-December, corresponding to $k = 8$.

[5]We here assume that these months have the same amount of days, in reality we have to multiply with a factor of 30/31, as July has more days than June.

3.5 Optimal Adjustment Function

3.5.1 Differences in Modeling Approaches

In the previous sections we show how the different adjustment functions considered in this thesis are linear functions of the observed Futures prices used in the calibration of the PFC. How the PFC then change with respect to a change in any of the Futures products used in the calibration is then independent of the current price level of said Futures products. Dependent on which model we consider, this derivative might be dependent on the number of products used in the calibration, and which periods these products cover. As we have this independence between the current level of all Futures products and the derivative, it seems natural to model the dependency of each single Futures product individually, and not as one curve. We will here look at how the different adjustment functions behave, and from this make conclusions on what is a natural basis for the adjustment curve. We will list the differences in the models and conclude which characteristics are good, or how we can improve the observed characteristics which are not fitting.

Functional form: In the model in Benth, Koekkebakker, and Ollmar, 2007 and the Novel model, we have a functional form for our adjustment function, while the method in Fleten and Lemming, 2003 model uses a dummy variable approach, weighting the different periods independent of some function. The biggest drawback with a functional form is that such a model has a typically large spillover effect, and no cut-off point. This means Futures prices observed for 2017 will affect the prices for all years modeled by the PFC. It also means that a curve covering three years might be different from the three first years of a curve covering four years. The upside with a functional form of the adjustment function is that we get a naturally smooth curve, which is not the case with a curve based on dummy variables. If one wants to use a curve based on a functional form, we would suggest using functions with a compact support. This means the function is no-negative for a compact set and otherwise zero. Typical examples are the Bump function:

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases},$$

A problem with this function, is that we can't analytically compute the integral, so we can't make a linear combination of such functions that will perfectly replicate the observed Futures products. Therefore a function of the form:

$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases},$$

might be more suitable. Both functions are positive for $|x| < 1$ and 0 otherwise. Both functions can be scaled for other periods. A combination of such functions can give the wanted spill-over effect.

Dummy Variables: As a contrast to functional form, one can use a model based on dummy variables, as is the case in the methods described in Fleten and Lemming, 2003 and Biegler-König and Pilz, 2015 where individual weights are assigned for each day. A combination of these two models can be used to gain more variability for different periods, while also providing the spillover-effect and smoothness obtained by using the smoothing approach from Fleten. We observe in the Fleten method that the spillover effect is approximately 0 after a certain point. To gain more control of the curve, one could add constraints setting the spillover-effect to exactly 0 after a certain point, this will not change the curve to a large extent.

One could also consider other basis functions as a starting point. Where the Fleten method shifts all prices by 1 euro/MWh when the price of a Futures contract change with 1 euro/MWh, if we set $\lambda = 0$. We could assume other ways to change the curve. One example would be to add more weights to June in the second quarters product than the other months, as this month is more affected by an increase of photovoltaics than April and May.

Another approach we consider is to chose a triangular basis function, as shown in Figure 3.7 at page 64. This results in a smoother transition between periods, as we do not get the sudden jump between periods, and we can therefore chose a smaller λ-value. A smoothed comparison between these versions is shown in Figure 3.8 at page 65. We get a higher peak with such a framework, and a larger spillover effect.

These are just suggestions for how to combine the smoothing approach from Fleten with an alternative shape. It is also clear from this that one can use the smoothing on the adjustment function only, if one is already satisfied with the smoothness of the seasonality curve. Then one does not suppress the weekly/daily seasonality, as we have observed earlier can be a problem with this method.

Dependence on marked granularity: The methods by Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 are independent of the market structure, and adjustable to any arrangement of observed Futures products. This leads to, as described earlier, a theoretical arbitrage for the model by Benth when we include new products to the optimization. We do not get this for the Fleten method, but we still observe an adjustment function that is dependent on the number of observed products. In this model, if we add products before the spillover-effect is cut-off, this will affect the adjustment function. When using an adjustment function as in the Novel approach, where the maximal number of products are taken into consideration from the beginning we do not observe this as we can not implement arbitrary constraints on this curve.

There is no right or wrong way to do this, but it seems reasonable to have a cut-off point, where everything after this point, does not affect the adjustment function. This can be obtained in the Fleten method, as mentioned earlier, by forcing every point on the curve, after a certain date, to be zero.

Linearity: In the models we have discussed there is a linear relationship between the Futures prices and the PFC, but from interest modeling there are examples of curves with a non-linear relationship, as discussed in Caldana, Fusai, and Roncoroni, 2017. In a non-linear framework we will have a dependence on the current level of the Futures prices, which adds complexity to the problem. Also, after a new product is observed in the market, there is a linear relationship between the Futures products, as the sum of three monthly products equal the quarterly product and so on, so a linear relationship seems natural even before we observe the individual monthly products, to preserve a continuity in the modeling framework before and after the introduction of a new product.

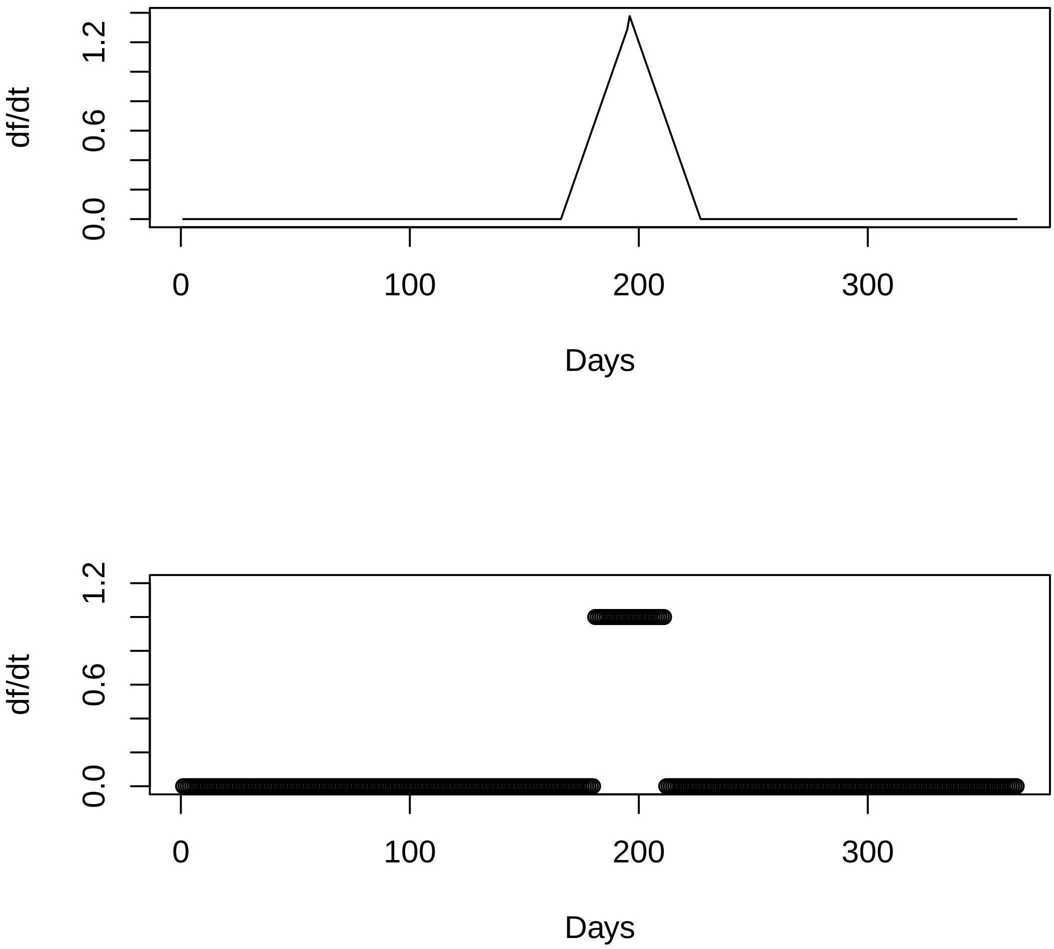

FIGURE 3.7: Showing how our new basis adjustment curve based on a triangle looks like compared to what is used in the method by Fleten

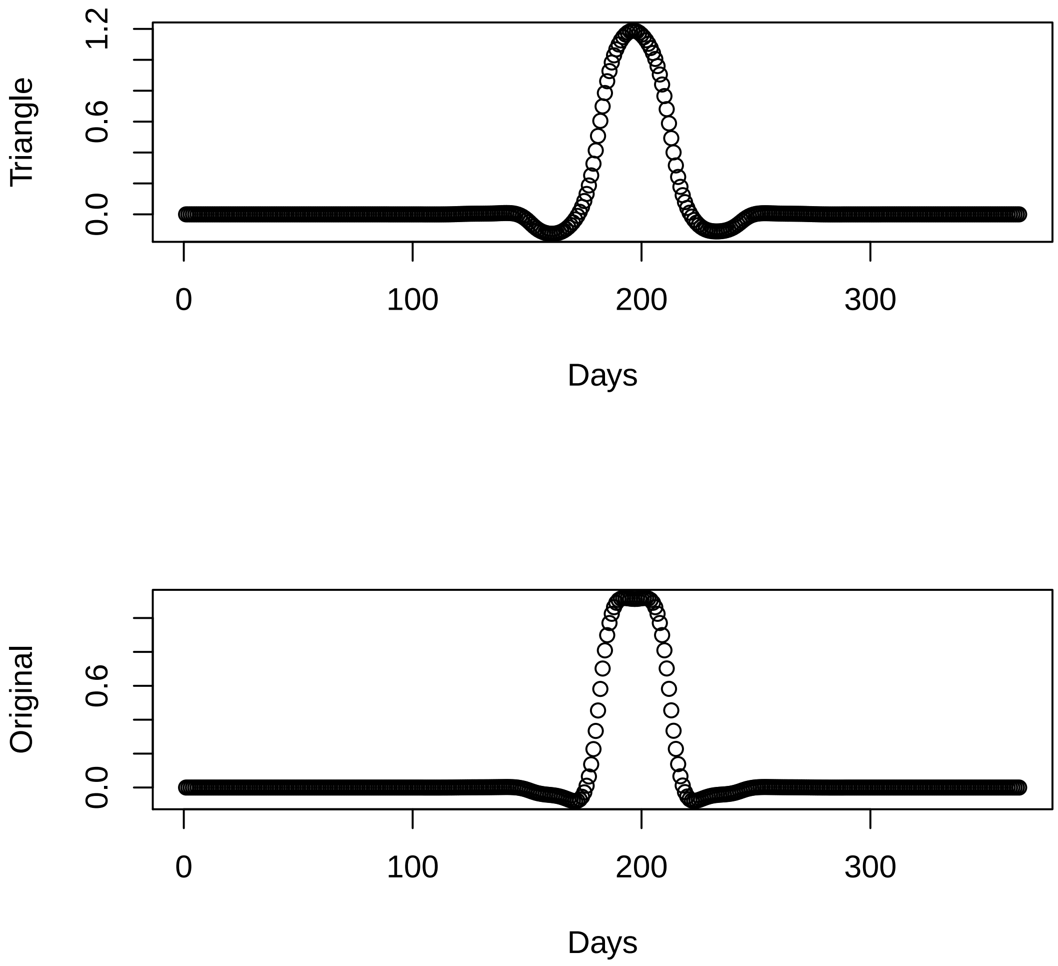

FIGURE 3.8: The two adjustment Functions (triangular and original) smoothed out with lambda=100.

3.5.2 Characteristics of the adjustment functions

In this section we will study the adjustment functions by looking at the graphs of what we have called the derivative of the adjustment functions, and from that explain the characteristics we have observed earlier. From that we will also explain what are natural characteristics of an adjustment function and explain how such a function can be constructed.

To study the characteristics of the adjustment functions by looking at the individual Futures products, can be a cumbersome task as there are so many different formulations considering which period you look at, and what other products are at sale at that moment. Because of this, it is nice to know if there are any similarities between these different formulations. By studying the different plots, we see that there are great similarities in the Novel method and the Fleten Method, but not in the Benth Method. This comes from the fact that the number of parameters are constant in the two first methods and not in the Benth method. Therefore, it is easier to study the characteristics of the Fleten and the Novel model as they are not that dependent on the number of Futures traded.

In Figure 3.9 at page 67 we see how the different adjustment functions change with respect to a change in the March Futures price, when we have different numbers of Futures products as input. As we see, in the method in Benth, Koekkebakker, and Ollmar, 2007, the shape of the adjustment function is highly dependent on the number of Futures products used in the construction.

In the two other methods, we observe the adjustment function is independent with respect to the number of products used in the construction. In the adjustment curve used in Benth, Koekkebakker, and Ollmar, 2007, the curve consisting of black dots are made from the three first monthly Futures, and one Futures product for the rest of the period. The blue curve is constructed with four monthly Futures products, purple with five, and so on. The curves flatten out in the long end when we introduce more Futures. This is natural as we get more periods where the total change needs to be equal to zero.

For the novel method we observe that the prices in March are heavily affected, with a peak in the middle of March, and that we have an effect on all other periods as well, but this effect is slowly dying out. The fact that the effect dies out comes from the fact we use a spline curve, with a normal trigonometric curve we would have had the same function for the whole period. The Fleten method change all prices in March, apart from at the edges by a similar amount, and have quite a small spillover-effect to the other months. We observe that the spillover effect converges to 0 relatively fast. What the Fleten method does, is in the general case, with no smoothing, to change each price in the relevant month by the same amount as the Futures product change with, also if the Futures product has a price change of 1 €, then the price of each day in that period change with 1 €, while all other prices are unchanged. When the smoothing factor increases, the change in the Futures prices have a greater spillover effect, as one want a smoother transition between months. One can also see that the derivative at the transition times between adjacent months seems small, meaning that the prices at these points are less variable than at the middle of a period.

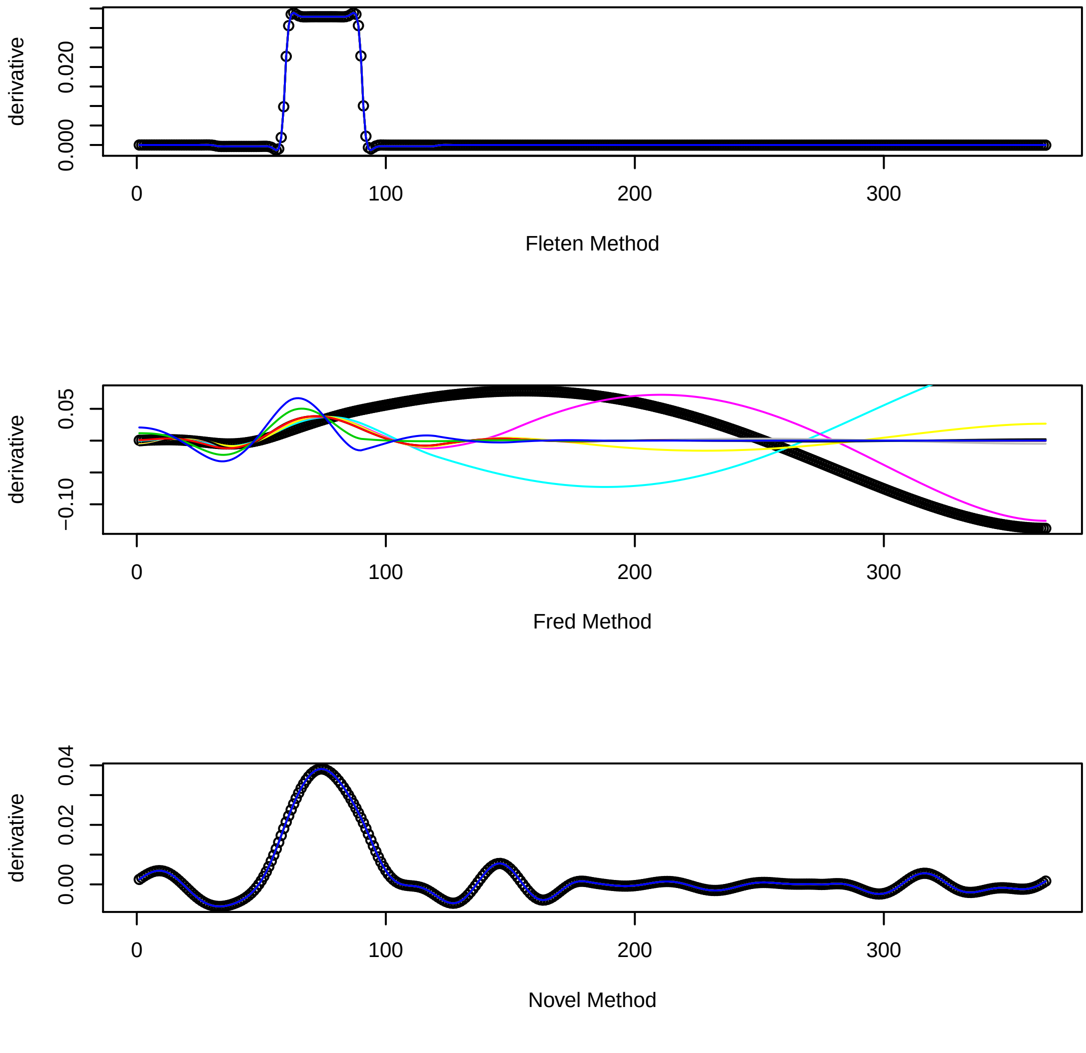

FIGURE 3.9: The derivative of the adjustment functions with respect to the March Future, where the number of Futures products used as input range from 3 to 12.

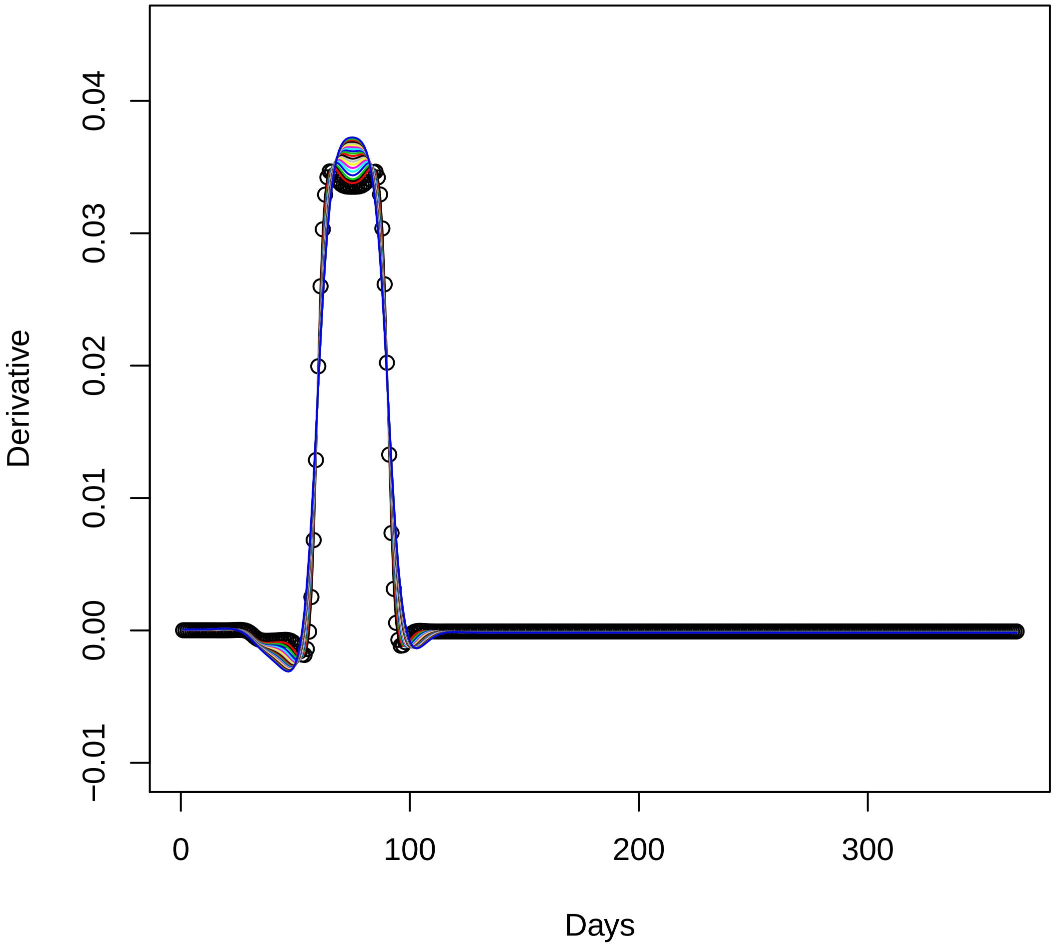

FIGURE 3.10: The change of the adjustment curve covering March when the λ smoothing parameter changes. λ-values between 10 and 200. The dots correspond to $\lambda = 10$, and when λ increases we observe increasingly higher values for the middle of March.

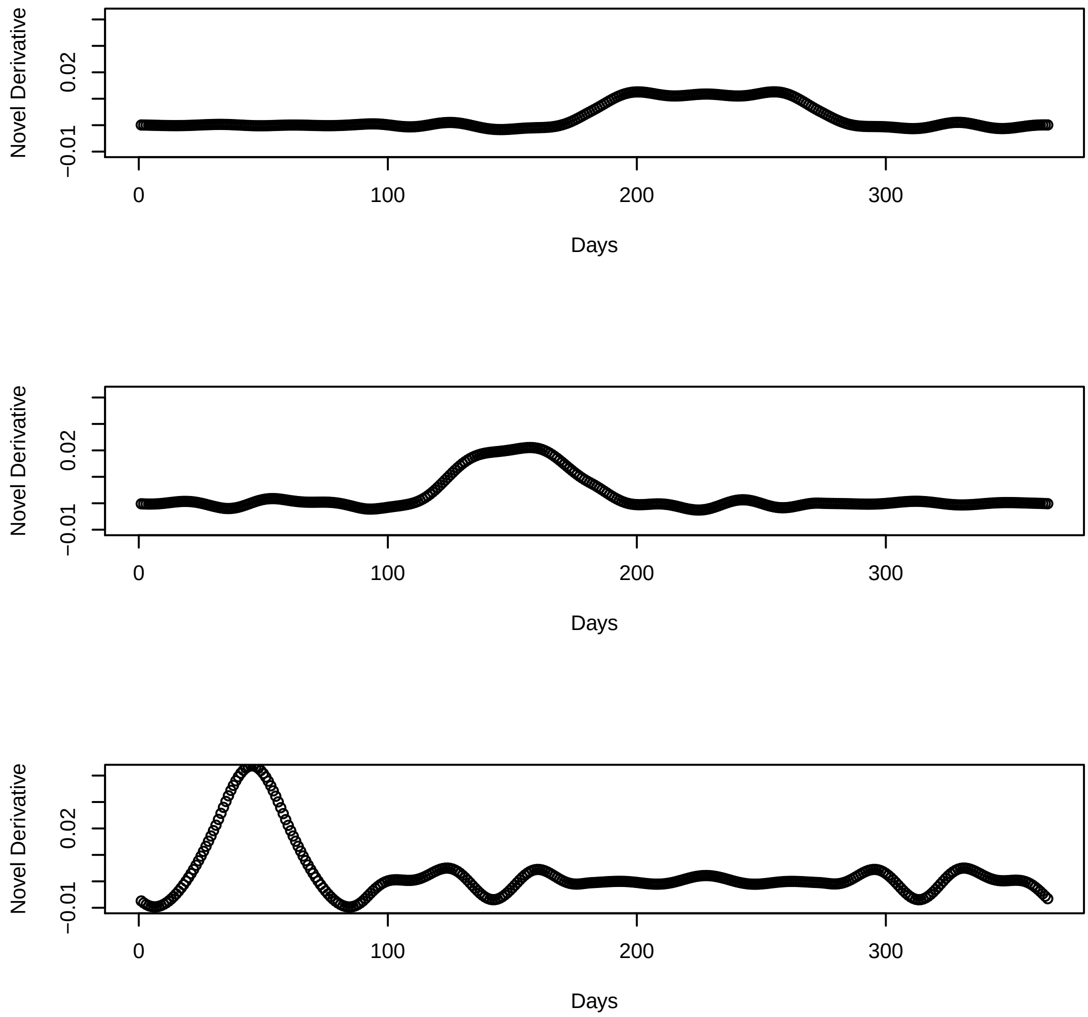

FIGURE 3.11: The Novel derivatives curve with respect to Jul-Sep Futures, May-June and February Futures respectively

In Figure 3.10 at page 68 we observe how the adjustment function changes when the λ-parameter increases. As the λ-parameter is increasing the curve gets increasingly smoother and the spillover-effect is increasing. We also observe that for a low λ-value, the curve is decreasing for the middle of March, while we see an increasing peak when λ is increasing. This comes from the fact that the increased spillover-effect for higher λ-values lead to a negative spillover-effect from the adjacent periods, and the increased peak is a response to this.

In Figure 3.11 at page 69 we show the Novel derivatives curve with respect to three different Futures product: July-September, May-June and February. The trend seems to be that the longer the period, the flatter the derivative becomes, with a lower peak, and the spillover effect for other months decrease. Despite this we see a lot of spillover effects in all three curves, which is natural as our trigonometric spline will never be equal to zero. The spline part will dampen this effect, but it will not dampen it entirely. It is not unheard-of that the February Futures price have an effect on the December prices, but it seems hard to estimate this effect, and therefore it seems more reasonable that this effect goes to zero in the long end of the curve.

Construction of Adjustment Function: To combine these characteristics to construct a new adjustment function, one can proceed as follow: First consider the Futures product with the shortest delivery period one will consider, we will call this the atomic Futures product. By first constructing the adjustment function for this product, which we call the atomic adjustment function, we can take linear combinations of such products to construct adjustment functions for products with longer delivery periods. By assuming the monthly Futures being the atomic products, a combination of three monthly Futures will give us the quarterly adjustment function, and so on. If one assumes all months in a quarter is equally sensitive with respect to the quarterly product, take the sum of the monthly products as the quarterly one. If one expects one monthly product is more sensitive, one weights this product more. The methods by Fleten and Benth are constructed without considering what type of Futures products that are traded. We would suggest to take that into consideration when constructing the curve, as you then avoid situations as the arbitrage possibility we have shown for the Benth method.

3.6 Conclusion

This section of the Thesis covers the relationship between our adjustment function and the observed Futures prices in three different models. In this section we do various observations of how the adjustment function in the different methods work, and from this we do conclusions based on personal opinions which of these characteristics are natural. As these conclusions are based on personal opinions, we would not say these conclusions are definite, but more guidelines and up for discussion. These conclusions are meant for a model which is linear in its parameters, which is the case for the discussed models. We will list the points we have discussed in this section, offering our own considerations on each point. At certain points we relate the discussion to what happens when the March Futures price changes, as we have previously focused on this product.

Uniformity: As a general rule it seems natural that the adjustment function behaves similar for all changes in all Futures products. Meaning that a change in the product covering March should induce a similar change as a change in the product covering December for said and adjacent periods. This can be altered on an individual basis, but that should be exceptions, and not the rule, and should have an economical argument backing it up. This is the case for the Fleten method, but not the Benth method, and the Novel method lays between these two.

Cutoff Point: As we have argued for, it is natural that a Futures product affects the adjacent periods, but it seems little reason for why the price of the March Future should affect the prices in December and vice versa. Therefore, it seems reasonable that there exists a cut-off point, where after this point said product do not influence the prices any more. Taking the March Future as example, a natural evolution for the adjustment function would be positive in the start, followed by negative until it hits a cut-off point where it has no more impact. Possibly followed by more such faces, where the amplitude is gradually decreasing. The exact placement of these points are up to discussion, and should be dependent on the length of the Futures product, as a yearly contract should have more spillover effect as a Monthly contract. Such an evolution we observe for the Fleten method, while for the two other methods we see that the March Future influences prices over the whole period.

Independent from number of Futures: It seems natural that how the March Futures product influences the other periods should be independent of how many products we see in the long term, and also to some degree on how many products we see on the short term [6]. We see this is the case with the Fleten method and the Novel method, while the Benth method is highly dependent on the number of products taken in the optimization, as this determines the number of parameters. A rule of thumb could be that the number of products outside the above mentioned cut-off point is irrelevant. Products in this period will of course make an impact, as the total spillover effect for such a period has to be 0.

Equal Uncertainty: The different days should show relatively equal uncertainty with respect to the Futures. So if one day is affected by a spillover effect from one period, this should result in this day is relatively less affected by what happens with the product covering this period. This we observe nicely in figure 3.10 where we see for a low λ-value (black dots), we have a smaller spillover effect, which only effect the beginning of the adjacent periods, resulting in slightly lower derivative for the middle of March, while for a high λ-value (blue line), we observe a negative spillover effect for the middle of the adjacent periods, meaning we need a higher derivative for the middle of March [7]. Such effects are harder to analyze for the other models, as each the spill over effect is much larger and does not have a cut-off point.

These beliefs about the spillover effect are personal beliefs, but they seem to offer a natural starting point. One might argument for other ways to model the spillover effect. The important thing is always to understand how the model works, and what characteristics it possesses. As an example Benth and Paraschiv, 2017 analyze a set of 2386 HPFCs for PHELIX, where they first construct curves with a five-year time horizon, with the Fleten method ,where $\lambda = 0$. These curves are then truncated to two years. According to our results, the truncation is not needed, as with a λ value of 0, there is no spillover effect, and one could directly construct curves with a two-year horizon resulting in the same set of curves.

Another reason for why one should understand this spillover effect is that the PFC is used by several parties, for different reasons inside one company, as explained in the paper by Biegler-König and Pilz, 2015. Understanding how the spillover effect works, makes it easier to make sure the PFC used by the different parties inside a company is the same, even if the number of Futures product taken into the construction differs. As an artificial example consider two traders inside the same firm, one is trading electricity for the current and next year, while the other trader trades electricity next year and the following three years. They will need different products for their construction, but as their curves overlap for one year, they will want the curves in this region to be equal. If the spillover effect is large, both traders will need a lot of products not directly influencing their time horizon. This might lead

[6] Products in the same year as said March product.

[7] The understanding is that the higher spillover effect also applies to the February and April products, which will drive the prices in the middle of March slightly down, resulting in a need for higher derivative during the middle of March then what is the case when February/April do not affect these prices.

to a problem for both traders. Trader 2 will need to frequently update his data set, as contracts on the short term of the curve are traded more liquidly than products on the long term. Another effect is that more contracts used in the construction results in a more data heavy estimation. A thorough understanding of the spillover effect leads to a more efficient construction, as all parties knows the exact number of Futures products needed in their relevant construction.

Chapter 4

Stochastic Model for the PFC

4.1 Introduction

In the previous sections of this thesis we have shown how the construction of a PFC combines the seasonal characteristics observed for electricity prices and the information of the current Futures prices to give a price of electricity for delivery periods down to one hour. We have also described how the constructed PFC changes when the prices of the observed Futures products change in three different models (Fleten and Lemming, 2003, Benth, Koekkebakker, and Ollmar, 2007 and a novel approach based on a constrained least squares approach of a model based on trigonometric splines). This means we can construct the PFC as it will be tomorrow, if we know what the Futures prices will be tomorrow.

In the same sense, it seems natural that if we have some estimate, or distribution for the Futures prices tomorrow, we could use this distribution to create a distribution for the PFC. As the PFC is a linear combination of the observed Futures products, this distribution will be a linear combination of the distribution of said Futures products.

In the literature, the modeling of electricity Futures is widely discussed, both with respect to economical characteristics from different markets, and how these characteristics can be modeled mathematically (see Lucia and Schwartz, 2002, Cartea and Figueroa, 2005, Benth, Kallsen, and Meyer-Brandis, 2007, Benth and Koekebakker, 2008, Benth et al., 2014 and Biagini, Bregman, and Meyer-Brandis, 2015). These papers focus on Futures contracts with fixed delivery periods. This is not the case for Futures products for electricity in the sense that today one might observe quarterly products while 3 months from now one might observe the individual monthly products. In this section we will propose a framework for modeling of Futures contract where the delivery periods are arbitrary, and where the current price is given by the constructed PFC. As the delivery periods are arbitrary, we can in this sense consider this a modeling framework for the PFC itself.

Benth and Koekebakker, 2008 consider the Futures contracts as swaps, as they pay the difference between a fixed rate, defined by the price of the Futures contract and a variable spot price over the relevant time period. In such a framework interest rates will have an impact on the modeling, depending on when the payments between the two parties are settled. In this section we will simplify this by assuming zero interest rates as interest rates will not give particular insight to our modeling approach. Benth and Koekebakker, 2008 consider a log-normal approach based on the HJM framework (Rutkowski and Musiela, 1998) where they model atomic swaps, and treat the other swap contracts as the sum of these atomic swaps. Shortcomings with such a modeling approach is that contracts with longer periods than the atomic swaps will in general not be log-normal, as a sum of log-normal distributions are in general not log-normally distributed. They also consider the monthly products as the atomic swaps, while in reality when coming close to delivery weekly and weekend contracts are also traded. As one can't find two or more independent random variables which is in sum equal to a log-normally distributed random variable, one can not find independent processes for the weekly contracts summing up to a log-normal monthly contract. It is clear that one could use products with smaller delivery periods as the atomic products, but that would lead to complicated distributions for the monthly products, quarterly and yearly products.

In Benth and Paraschiv, 2017 they construct a data set of 2386 HPFC for the PHELIX and do a statistical analysis of this data set. There they conclude that the risk premium varies around zero and may be both positive and negative depending on the risk aversion in the market, as is consistent with the literature.They also observe a Samuelson effect, where the front month and front quarter contract is more volatile than the remaining contracts. In Benth and Paraschiv, 2017 they consider this data set as a random field and propose a spatio-temporal random field approach for the modeling of the forward prices. We will also propose a stochastic model for the forward prices, but our main goal is that this model should be consistent to the construction of the PFC.

We want to construct a stochastic model for the forward prices for electricity such that we can find a distribution for our PFC at all times $t = T$ in the future. If we today at time $t = 0$ simulate the PFC for time $t = T$, we can also average out this simulated PFC over the periods corresponding to a set of Futures products, to get simulated Futures prices for these periods at time $t = T$. Our goal is that for all realizations of our simulated PFC, the PFC constructed using this set of estimated Futures as input should equal the simulated PFC. Such a framework is to our knowledge new in the literature as most studies constructing stochastic models for the Futures products do not consider the mathematical relationship between the PFC and these Futures products.

By constructing an HPFC for electricity, one can price load curves requesting the delivery of electricity at certain delivery periods in the future. While this is of course is highly useful for participants trading in electricity, this only gives us the fundamentals for energy trading as this tells us nothing about the uncertainty of this price. Having some understanding of the uncertainty makes it possible to choose between trading now, or at some later point in time, and also gives the possibility to trade in derivatives of the forward curve. As this sort of analyses are thoroughly studied for electricity Futures, we will here discuss how these characteristic are transferred to the PFC by investigating the linear relationship between the PFC and the Futures products shown earlier in this thesis.

The background for this section is the linear relationship between the Futures prices and the PFC shown in the previous section. We will investigate what conditions we need on the Futures products to get a reasonable model for an arbitrary load curve priced from the PFC.

The motivation for the need of load curves with arbitrary length is that within a company which produces and sells electricity the PFC is used by several parties within said company (Biegler-König and Pilz, 2015). Each individual party will price products with different delivery periods and will use the PFC for different purposes. Having Futures prices that can be split up into arbitrary lengths assures that these parties within the company can price their products without knowing exactly what products the other parties in the company trades with, and still keep the stochastic models of the different parties consistent to each other.

As an example, consider a company with two traders, one trading monthly products and the other trading in weekly products. Assuming independent log-normal models for the different products it is clear that one can not separate the monthly products into four weekly products and a product covering the remaining days. This is because the log-normal distribution is not divisible, and the distribution of the sum of the individual monthly products will differ from the distribution of the monthly product. Correspondingly, any risk measure on the sum of the decomposed monthly product will differ from the risk measure on the monthly product. On the other hand assuming the weekly products as the atomic products leads to a problem as the months do not divide perfectly into weekly products. There is also the problem that the sum of log-normal distribution do not have a known distribution making the sum of such distributions hard to handle. A common approach to this problem is to approximate the distribution by a skewed log-normal distribution (Hcine and Bouallegue, 2015). Basket options, where the underlying is the sum of two or more risky assets, are well studied in the literature, and there are several approaches for finding approximate solutions to such options, both for the log-normal and the more general log-Lévy type of processes (Milevsky and Posner, 1998, Brigo et al., 2004 and Xu and Zheng, 2010).

We will investigate how a model driven by Lévy-processes, and not exponential Lévy-processes, will behave, since we are then working with infinitely divisible processes, and we can then split Futures contracts into independent products with shorter delivery periods as needed. We will investigate what shortcomings such processes have, and we will investigate which Lévy processes are viable and how the construction of our PFC affects the parameters of the model.

The rest of this section of the thesis is organized as follows: In Section 4.2 we introduce our problem and state and justify the stochastic model we will use in this section. In Section 4.3 we investigate the relationship between our Futures products and the resulting PFC to observe how the parameters need to compare to each other in the different models. In Section 4.4 we expand our model to saying the starting price coming from the PFC has some uncertainty, and we observe how this affects our model. In Section 4.5 we formulate our full model, as well as explaining what changes in the model when the granularity of the observed products change. In Section 4.6 we conclude.

4.2 Introduction of Problem

The problem we want to discuss here is how to construct a stochastic model for load curves with arbitrary delivery lengths which is consistent with how we construct our PFC. If we at day j observe a set of Futures products denoted by:

$$V_j^n = \{v_{\Delta_1}^j, ..., v_{\Delta_n}^j\},$$

then we have earlier shown that the price of the PFC with delivery at day i is given by:

$$PFC_j(i) = \tilde{s}_j(i) + \sum_{k=1}^{n} d_{j,i}^k v_{\delta_k}^j.$$

Here $\tilde{s}_j(i)$ is the seasonality normalized with respect to the Futures prices, this means:

$$\sum_{i \in \delta_k} s_j(i) = v_{\delta_k}^j.$$

$d_{j,i}^k$ is the effect futures product k has on electricity with delivery on day i. If we want to look at the uncertainty of this price at some day j_2 in the future, we have to possibilities:

Alternative 1: The number of products observed at day j_2 is equal to the number observed today, and the stochastic model for the PFC will be the linear combination of these products.
Alternative 2: The number of products observed at day j_2 is not equal to the number of products observed at day j, as a result of cascading of the Futures products. Then the stochastic model for the PFC will be a linear combination of Futures products that are currently not observable.

Considering alternative 2, our construction method for the PFC gives us a starting point for the unobserved product, as we get an implied price for each day. What we will investigate in this part is if we have certain characteristics for the already observed Futures products, like volatility, Samuelson effect, skewness and mean reversion, can these characteristics be transferred to the implied products by the manner in which we construct our PFC.

4.2.1 Build-up of the Model

We will in the following denote our Futures product covering the period $[T_i^s, T_i^e)$ by

$$F(t, T_i^s, T_i^e), \tag{4.1}$$

were t is the day the Futures price is observed at. At $t = 0$ (today), $F(0, T_i^s, T_i^e)$ is observed, and for $t > 0$ we assume $F(t, T_i^s, T_i^e)$ is given by some distribution determined by the stochastic process used for the modeling of our Futures products. If we assume that at the time-point T_1 we observe a new product splitting the time period $[T_i^s, T_i^e)$, into the periods $[T_i^s, T)$ and $[T, T_i^e)$. Then we know that for $t < T_1$ our price for the implied product:

$$F(t, T_i^s, T),$$

will be calculated as a linear combination of the products observed at time t. While for $t \geq T_1$ this is a traded product at the exchange, and we want to be able to model this independently from the other observed products. As we want processes that posses this divisibility property, we will work with processes driven by Lévy processes, as these processes are infinitely divisible.

We will in the following assume we have a set of atomic Futures products, which are the products with the smallest delivery period one can observe. We will not state explicitly what delivery periods these products have, as we want to make a general framework where one can use products with arbitrary length as atomic products, but to help visualize the problem one can think of these products as monthly products.

We will assume these atomic products are modeled by Ornstein-Uhlenbeck processes (OU-processes) where the noise is driven by a Lévy process. The choice of an OU-process is natural as we want to incorporate a mean-reverting effect, which is dependent on the risk-premium of the Futures prices. We also want to allow for time dependent volatility processes as we observe more variation as we come closer to maturity, so we have a time dependent volatility function.

$$dF(t, T_j^s, T_j^e) = a(m - F(t, T_j^s, T_j^e))dt + \sigma(t, T_j^s, T_j^e)dL_t; F(0, T_j^s, T_j^e) = F_0, \tag{4.2}$$

where L_t is a general Lévy process (Tankov, 2003, Øksendal and Sulem, 2005 and Papapantoleon, 2008). We will mainly focus our attention on the case where L_t is a standard Brownian motion, but we will briefly discuss other possible Lévy-processes and their limitations. We also assume that the parameters of our atomic processes are the same before and after we observe this product in the market.

To shorten notation we will write $F_j(t)$ and $\sigma_j(t)$, and rather let the subscript j denote which time period $[T_j^s, T_j^e)$ we work with. The typical form for the volatility function in the literature, when considering one delivery point, is

$$\sigma_j(t) = \sigma_{0,j} \exp(\kappa_j(T_j - t)),$$

where T_j is delivery time of the Future. As we are here not working with a traditional Futures product, but a swap paying the difference between the Futures price $F_j(t)$ and the variable spot price over the corresponding time period, we don't have one specific delivery point. Candidates for the delivery point will then be the start, end or middle point for the delivery period. This choice is in some sense arbitrary, since we can scale the variance by the $\sigma_{0,j}$ parameter.

For the price forward curve, we will use the notation:

$$f_t(i, V_t^n), \tag{4.3}$$

which is the average price of electricity with delivery at day i in the future, as seen from day t, with the set of Futures products denoted by V_t^n. As we have shown earlier, the change in the forward price with respect to the Futures products is linear, and defined as:

$$f_{t+1}(i, V_{t+1}^n) - f_t(i, V_t^n) = \sum_{j=1}^{n} d_{j,i}^n (F(t+1, T_j^s, T_j^e) - F(t, T_j^s, T_j^e)). \tag{4.4}$$

Where $d_{i,j}^n$ is the derivative of our PFC for day i with respect to product j. As earlier n determines the set of products we observe, if that is relevant for the derivative. Equation (4.4) is for a discrete model, we will in the following assume a continuous model. As

$$\frac{\partial d_{i,j}^n}{\partial t} = 0,$$

we get correspondingly

$$df_t(i, V_t^n) = \sum_{j=1}^{n} d_{j,i}^n dF(t, T_j^s, T_j^e), \tag{4.5}$$

when the number of Futures products is constant, and we take the derivative with respect to t.

In reality the number of Futures products is not constant for all time-points t, as the granularity of observed products is increasing as we come closer to delivery. Because of this we will need a consistent way to take the Futures products into account, even when we assume the granularity of these can change. As we have discussed earlier, it is natural that when a Futures product is cascading into two or more products, that if the PFC correctly estimates these products, the PFC should not be affected by the cascading. This is the case for our Novel method, and also partly for the method proposed in Fleten and Lemming, 2003. This characteristic is not observed in the method by Benth, Koekkebakker, and Ollmar, 2007, since the requirement for more parameters in this method when we have more constraints lead to a deterministic shift in the curve when adding said constraints to the optimization.

The goal of our research is to show how forwards products of arbitrary delivery length can be modeled in one framework, when the pricing is consistent with the construction of the PFC. To ease the visualization we will consider 12 monthly Futures

$$F(t, T_j^s, T_j^e),\ j \in [1, 12],$$

where at any time t we observe a linear combination of these. As an example one can observe the quarter Futures product covering October-December, but not each individual monthly product:

$$\bar{F}(t, T_{10}^s, T_{12}^e) = \sum_{k=10}^{12} b_k F(t, T_k^s, T_k^e).$$

Here we observe the left side, but does not know the individual weights b_k. From the construction of the PFC we get an estimate for $F(t, T_k^s, T_k^e)$, given by

$$F(t, T_k^s, T_k^e) = s_k + \sum_{j \in J} a_j F(t, T_j^s, T_j^e)),$$

where J is the set of observed Futures products at time t. The value a_j is the sum of the values $d_{i,j}^n$, we will in the following use a_j when we talk about longer periods, and $d_{i,j}^n$ when talking about days.

Our goal is twofold. First we want to see how the parameters driving the individual monthly products will compare to each other, given how we construct the PFC. The second goal will be to see how the expectation and variance of these models will differ before and after a new product is added to the market.

We will in the following generally work with a simplified model where we either have two or three products. By this we will observe how one product can be split into two new ones, and of one of these two products can be split into another new product. We want to figure out how the parameters of the individual processes compare to each other, assuming we keep the linear relationship we have shown earlier. We will show that when working with OU-processes the parameters needs to be the comparable for both processes in such a way that the sum is again an OU-process. Therefore we can conclude that if the individual monthly products are OU-processes, the weighted sum of these giving the quarterly product will also be an OU-process.

4.2.2 Sum of Ornstein-Uhlenbeck processes

We have earlier explained that we are interested in what happens when Futures products are cascading, and how we can model this with OU-processes. So we want to split up a sum of processes into its individual parts. To understand how this can be done, one should first understand what happens when we add two, or more, OU-processes to construct a new OU-process.

In the following, we will assume we have two processes $F_1(t), F_2(t)$, both following these dynamics

$$dF_i(t) = \alpha_i(m_i - F_i(t))dt + \sigma_i(t)dB_i(t);\; F_i(0) = F_0^i.$$

We are then interested in the linear combination:

$$F(t) = c_1F_1(t) + c_2F_2(t);\; F(0) = c_1F_0^1 + c_2F_0^2$$

where c_i is a constant corresponding to the length of the corresponding product. For example: If we have one Futures product F cowering one year, and want to price two new products F_1 and F_2 cowering the first quarter and the remaining three quarters, then $c_1 = \frac{90}{365}$ and $c_2 = \frac{275}{365}$.

Given this, the dynamics of $F(t)$ is:

$$\begin{aligned} dF(t) &= (c_1m_1\alpha_1 + c_2m_2\alpha_2 - c_1\alpha_1X_1(t) - c_2\alpha_2X_2(t))dt \\ &\quad + c_1\sigma_1(t)dB_1(t) + c_2\sigma_2(t)dB_2(t) \end{aligned}$$

This is an Ornstein-Uhlenbeck process if and only if $\alpha_1 = \alpha_2 = \alpha$, as we can do the substitution:

$$c_1\alpha_1F_1(t) + c_2\alpha_2F_2(t) = \alpha(c_1F_1(t) + c_2F_2(t)) = \alpha F(t)$$

If the mean reversion speeds are not equal we still have the closed form solution to $F(t)$, given by:

$$\begin{aligned} F(t) &= c_1F_1(t) + c_2F_2(t) \\ &= c_1F_0^1e^{-\alpha_1t} + c_2F_0^2e^{-\alpha_2t} \\ &\quad + c_1(1 - e^{-\alpha_1t})m_1 \\ &\quad + c_2(1 - e^{-\alpha_2t})m_2 \\ &\quad + c_1e^{-\alpha_1t}\int_0^t e^{\alpha_1s}\sigma_1(s)dB_1(s) \\ &\quad + c_2e^{-\alpha_2t}\int_0^t e^{\alpha_2s}\sigma_2(s)dB_2(s). \end{aligned}$$

This process resemble the classic Ornstein-Uhlenbeck process, as it has normally distributed increments, and it will revert towards some long-term mean. The difference is that while if a normal Ornstein-Uhlenbeck process equals this long-term-mean at some point t, we expect it to stand still, in the sense that the expected value of this process in the future, is equal to its current value. This is not necessarily true for the sum of two such processes. Therefore, we can expect it on a short term basis to change, but on a long term basis we expect it to move back to the present value.

This comes from the fact that the sum of the two processes can be equal to the expected price, even if the two processes themselves don't hit their respective means. Therefore one can expect the price to increase, decrease or stay put. We have that he expected value of $F(t)$ is:

$$\begin{aligned} E[F(t)] &= c_1(F_0^1 - m_1)e^{-\alpha_1 t} + c_1 m_1 \\ &\quad + c_2(F_0^2 - m_2)e^{-\alpha_2 t} + c_2 m_2 \end{aligned}$$

Is we assume the process starts in the long-term mean, meaning:

$$F(0) = c_1 F_0^1 + c_2 F_0^2 = c_1 m_1 + c_2 m_2,$$

but we assume our processes $F_1(t)$ and $F_2(t)$ do not start in their respected long-term means, for example:

$$F_0^1 > m_1$$

$$F_0^2 < m_2$$

and we assume taht

$$\alpha_2 > \alpha_1.$$

This gives us:

$$\begin{aligned} E[F(t)] &= c_1(F_0^1 - m_1)e^{-\alpha_1 t} + c_1 m_1 \\ &\quad + c_2(F_0^2 - m_2)e^{-\alpha_2 t} + c_2 m_2 \\ &> (c_1(F_0^1 - m_1) + c_2(F_0^2 - m_2))e^{-\alpha_2 t} + c_1 m_1 + c_2 m_2 \\ &= c_1 m_1 + c_2 m_2. \end{aligned}$$

This means that even if the process starts in its long-term mean, one expects the price to increase short term, as the value of $F_2(t)$ will increase faster than $F_1(t)$ will decrease. Similar situations can occur with other choices of the parameters. By including more than two different OU-processes the situation becomes even more complex.

A problem with this is that in electricity markets, one does not always observe all the Futures products, but one observes some linear combination of these products. If one estimates a starting price for the different products by the PFC, one will get different expected value of the sum of these products $F(t)$ depending on the price of the individual products coming from the PFC. For this reason having the same mean-reversion speed for all atomic products is a practical modeling assumption. In the following we will show that given how we construct our PFC, the only reasonable choice is to keep the mean reversion speed equal for all atomic products.

4.2.3 Economical beliefs of Futures Model

We use the term buying a Futures product, while in reality this is not the correct term. When talking about buying a Futures product, we are in reality taking about entering a Futures position, with multiple delivery points, so it can also be considered as a swap (Benth and Koekebakker, 2008). So by entering this contract one promises to pay the difference between the variable spot price and the pre-determined Futures price. We will in the following assume that there is no interest rate, so the point of time when the actual payment is done is irrelevant. We can therefore assume we pay for the contract when we enter it, and we will use the term buying a Futures contract instead of entering a Futures position. Setting the interest rate to 0 also simplifies the calculations, but the results are also valid with a deterministic interest rate.

In our model we use three parameters, the long-term mean m, the mean-reversion speed α and the time dependent volatility $\sigma(t)$. If we consider a general Lévy process, we also get more parameters which can model skewness or jumps of the Futures prices, but we will in mostly work with a standard Brownian motion. Benth and Paraschiv, 2017 give a thorough explanation of how the risk premium and Samuelson effect is for electricity Futures, both in existing literature, and in their model. They conclude that the risk premium can be negative or positive depending on the average risk aversion in the market. In general producers of conventional power plants (in particular nuclear power plant where marginal costs are close to zero), will accept all prices higher than their marginal costs in the Futures market, to make sure they can run their plant with a profit. With the increasing in-feed of renewables in the market, they might accept prices lower than the expected spot prices to avoid selling with a loss in the spot market. Equivalently a production plant consuming large amounts of electricity will accept prices higher than expected spot prices, as they want to secure a price which makes their total production profitable. This is just to illustrate why both positive and negative risk premiums can be seen for electricity. We will not assume either positive or negative risk premium, just allow for both to be present in the model. Benth and Paraschiv, 2017 also analyze the volatility structure, arguing that the front month and quarter Futures show more volatility than the remaining products.

In the following we will assume we a set of parameters (m, α and $\sigma(t)$) for the stochastic processes driving the observed Futures products. We will then investigate that if given how we construct our PFC, can we get an implied set of parameters for the unobserved Futures products in the same sense that we get implied prices for these objects from the PFC. If we get such a relationship, this means we can not construct stochastic models for the Futures products independent of the construction method for the PFC. We will assume in the following that the stochastic model for a certain Futures product, does not change when this product is cascading into two or more independent products, if we correctly price these products from the PFC. If we do not correctly price these products, it is a signal that there is something wrong in our model, and we might want to readjust the parameters. We will also show how it is natural to readjust these parameters.

4.3 Model Setup

The aim of this section is to find a stochastic model for an arbitrary Futures product from the PFC, given a set of Futures products and a linear relationship between said Futures products and the PFC. We will start with a simplified model, where we assume we observe a quarterly product and the first monthly product of said quarter. From the arbitrage free property we get two disjoint Futures products, one covering the first month, and one covering the two last months, we will denote these by F_1 and F_2. The prices here are the average prices for the corresponding period, so to get the price of the whole quarter we need to take the weighted average of these two products, where the weights correspond to the length of the delivery periods. From the construction of the PFC we get an implied price for the second and third month Futures, which we will denote $\tilde{F}_2^1$ and $\tilde{F}_2^2$, given by

$$\tilde{F}_2^j = s_2^j + a_1^{2,j} F_1 + a_2^{2,j} F_2.$$

As the weighted average of the two monthly products needs to equal F_2, we obtain this equation

$$c_1 F_2^1 + c_2 F_2^2 = F_2, \forall F_1, F_2 \in \mathbb{R}$$

This gives

$$\begin{aligned} c_1 s_2^1 + c_2 s_2^2 &= 0 \\ c_1 a_1^{2,1} + c_2 a_1^{2,2} &= 0 \\ c_1 a_2^{2,1} + c_2 a_2^{2,2} &= 1 \end{aligned}$$

where $0 < c_j a_2^{2,j} < 1$. We will in the following use the notation F_i interchangeably for the average price of electricity for the period cowered by this product, and we will also use F_i when we talk about the specific product. What we mean in each instance will be clear from context.

We will in the following denote Futures products in three ways: We have the whole period, for us corresponding to one year, the average price of this period will be denoted by F. This product we will then split into two or more products, where the price of each of these products will be denoted F_i. Each F_i can again be split up into two or more products, and the price of these will be denoted F_i^j. Correspondingly we will in the same sense use s, s_i and s_i^j. We will use the notation $a_k^{i,j}$ to denote the effect Futures product F_k has on the implied product F_i^j. We can then price F_2^j in two ways, either as earlier

$$\tilde{F}_2^j = s_2^j + a_1^{2,j} F_1 + a_2^{2,j} F_2$$

or when we only observe F, we get

$$\tilde{F}_i^j = s_2^j + a^{i,j} F.$$

As we also have:

$$\tilde{F}_i = s_i + a_i F$$

and as our adjustment curve should be arbitrage free to the observed number of Futures product, we get that these equations should coincide, giving us

$$s_i^j + a^{i,j} F = s_i^j + a_1^{i,j}(s_1 + a_1 F) + a_2^{i,j}(s_2 + a_2 F). \tag{4.6}$$

Since this should hold for all F, we get

$$a_1^{i,j} s_1 + a_2^{i,j} s_2 = 0 \tag{4.7}$$

and

$$a_1^{i,j} a_1 + a_2^{i,j} a_2 = a^{i,j}. \tag{4.8}$$

By working in such a simplified framework we implement our two most important characteristics:

1: How is the stochastic model of a Futures product which covers a sub-period of a currently traded Futures product affected by a price change in said product.
2: How is the model of this sub-period product affected by a product that is covering a different period than said sub-product.

It should be noted that we work with a normalized seasonality curve, averaging out to 0 for the periods corresponding to a Futures product. We could define it equivalently by looking at the residual Futures prices, defined as the difference between the Futures price and the average of the seasonality curve for the corresponding period.

4.3.1 Modeling Framework

We assume a model where the monthly products are the atomic products, so each individual monthly product is modeled by an OU-process as given earlier, denoted by:

$$\begin{aligned} dF_1(t) &= \alpha_1(m_1 - F_1(t))dt + \sigma_1(t)dB_1(t), & (4.9)\\ dF_2^1(t) &= \alpha_2^1(m_2^1 - F_2^1(t))dt + \sigma_2^1(t)dB_2^1(t), & (4.10)\\ dF_2^2(t) &= \alpha_2^2(m_2^2 - F_2^2(t))dt + \sigma_2^2(t)dB_2^1(t). & (4.11) \end{aligned}$$

We only observe F_1 and F_2, so our starting points are

$$F_1(0) = F_1,$$

$$F_2^1(0) = s_2^1 + a_1^{2,1} F_1 + a_2^{2,1} F_2,$$

$$F_2^2(0) = s_2^2 + a_1^{2,2} F_1 + a_2^{2,2} F_2.$$

Where we first assume that all Brownian motions $B_i(t)$ are independent. From this we get two expressions for the price of $F_2^i(t)$. On the one hand we would expect it to follow the path of the OU-process defined here, while on the other hand we know that in the Future, before we actually observe this project, we will price $\tilde{F}_2^i(t)$ as the linear combination of $F_1(t)$ and $F_2(t)$. In the following we will show which assumptions on the model are needed for these two definitions of $F_2^i(t)$ coincide.

We assume that for $0 \le t < T_1$ we observe the products $F_1(t)$ and $F_2(t)$, while for $T_1 \le t < T$ we observe the individual products $F_1(t)$, $F_2^1(t)$ and $F_2^2(t)$. Our aim is then to show what the distribution for these products are before and after T_1. We will also show how the parameters of the different products compare to each other, given the linear pricing rule we use for our PFC.

The solution to our stochastic differential equations (4.9)-(4.11), are of the form:

$$F(t) = (F(0) - m)e^{-\alpha t} + m + e^{-\alpha t} \int_0^t e^{\alpha s} \sigma(s) dB(s) \tag{4.12}$$

If we look specifically at the solution for $F_2^1(t)$, we have on the one hand the solution

$$F_2^1(t) = (s_2^1 + a_1^{2,1}F_1 + a_2^{2,1}F_2 - m_2^1)e^{-\alpha_2^1 t} + m_2^1 + e^{-\alpha_2^1 t}\int_0^t e^{\alpha_2^1 s}\sigma_2^1(s)dB_2^1(s). \quad (4.13)$$

This corresponds to the dynamic if we had observed today the starting price $F_2^1 = s_1 + a_1^1 F_1 + a_2^1 F_2$. On the other hand, for $t < T_1$ we only observe F_1 and F_2, and we will price $F_2^1(t)$ by $\tilde{F}_2^1(t)$ defined as:

$$\begin{aligned}
\tilde{F}_2^1(t) &= s_2^1 + a_1^{2,1}F_1(t) + a_2^{2,1}F_2(t) \\
&= s_2^1 + a_1^{2,1}F_1(t) + a_2^{2,1}\left(c_1F_2^1(t) + c_2F_2^2(t)\right) \\
&= s_2^1 + a_1^{2,1}\left[(F_1 - m_1)e^{-\alpha_1 t} + m_1 + e^{-\alpha_1 t}\int_0^t e^{\alpha_1 s}\sigma_1(s)dB_1(s)\right] \\
&\quad + a_2^{2,1}c_1\left[(F_2^1 - m_2^1)e^{-\alpha_2^1 t} + m_2^1 + e^{-\alpha_2^1 t}\int_0^t e^{\alpha_2^1 s}\sigma_2^1(s)dB_2^1(s)\right] \\
&\quad + a_2^{2,1}c_2\left[(F_2^2 - m_2^2)e^{-\alpha_2^2 t} + m_2^2 + e^{-\alpha_2^2 t}\int_0^t e^{\alpha_2^2 s}\sigma_2^2(s)dB_2^2(s)\right].
\end{aligned}$$

A natural requirement for our processes is that:

$$E[\tilde{F}_2^1(t)] = E[F_2^1(t)];\ \forall 0 \le t < T_1$$

as we expect our PFC to correctly price products that are not observed. We do not necessarily want these processes to have the same variance, as we in $\tilde{F}_2^1(t)$ average out our uncertainty, making the total variance smaller. This gives us the equation:

$$\begin{aligned}
E[F_2^1(t)] &= \left(s_2^1 + a_1^{2,1}F_1 + a_2^{2,1}(c_1F_2^1 + c_2F_2^2) - m_2^1\right)e^{-\alpha_1 t} + m_2^1 \\
&= E[\tilde{F}_2^1(t)] \\
&= s_2^1 + a_1^{2,1}\left((F_1 - m_1)e^{-\alpha_1 t} + m_1\right) + a_2^{2,1}c_1\left((F_2^1 - m_2^1)e^{-\alpha_2^1 t} + m_2^1\right) \\
&\quad + a_2^{2,1}c_2\left((F_2^2 - m_2^2)e^{-\alpha_2^2 t} + m_2^2\right). \qquad (4.14)
\end{aligned}$$

This is an exponential equation, with different exponents on the form:

$$c + \sum_{i=1}^{n} b_i e^{-\kappa_i t} = 0;\ 0 \le t \le T, \quad (4.15)$$

where we say $T > 0$ and we assume $\kappa_n > \kappa_{n+1}$. This has the solution $c = 0$ and $b_i = 0;\ i = 1, ..., n$. This means we can separate the terms dependent on t and the terms not dependent on t in (4.14). By doing the same considerations for $F_2^2(t)$ we get these equations:

$$m_2^1 = s_2^1 + a_1^{2,1}m_1 + a_2^{2,1}c_1m_2^1 + a_2^{2,1}c_2m_2^2 \quad (4.16)$$

$$m_2^2 = s_2^2 + a_1^2 m_1 + a_2^2 c_1 m_2^1 + a_2^2 c_2 m_2^2 \quad (4.17)$$

$$\left(F_2^1 - m_2^1\right)e^{-\alpha_1 t} = a_1^{2,1}(F_1 - m_1)e^{-\alpha_1 t} + a_2^{2,1}c_1(F_2^1 - m_2^1)e^{-\alpha_2^1 t} + a_2^{2,1}c_2(F_2^2 - m_2^2)e^{-\alpha_2^2 t} \quad (4.18)$$

$$\left(F_2^2 - m_2^2\right) e^{-\alpha_1 t} = a_1^{2,2}(F_1 - m_1)e^{-\alpha_1 t} + a_2^{2,2} c_1 (F_2^1 - m_2^1) e^{-\alpha_2^1 t} + a_2^{2,2} c_2 (F_2^2 - m_2^2) e^{-\alpha_2^2 t} \tag{4.19}$$

where the second set of equations can be split up more depending on whether the mean reversion rates α are equal to each other or not. We will first look at equation (4.16) and (4.17), by subtracting $F_2^i = a_1^{2,i} F_1 + a_2^{2,i} F_2$, we get:

$$(m_2^i - F_2^i) = a_1^{2,i}(m_1 - F_1) + a_2^{2,i}(c_1 m_2^1 + c_2 m_2^2 - F_2) \tag{4.20}$$

From this we get that the difference between the starting point of our processes, denoted F_2^i and the long-term mean m_2^i is dependent on the difference between the long-term mean of $F_2(t)$, which we can denote by $m_2 = c_1 m_2^1 + c_2 m_2^2$ and the currently observed price F_2, and the difference between the long-term mean m_1 of $F_1(t)$ and its starting point F_1. If $a_1^{2,1} > 0$, then $a_1^{2,2} < 0$, as we have $c_1 a_1^{2,1} + c_2 a_1^{2,2} = 0$, so the differences $m_2^1 - F_2^1$ and $m_2^2 - F_2^2$ are differently affected by $m_1 - F_1$. From this we can theoretically have that F_2^1 has a starting price lower than the long-term mean, while F_2^2 starts over the long-term mean. We saw earlier, with such starting points and different mean reversion rates for the different processes, the expected price of $F_2(t)$ might be non-intuitive with respect to its starting point and long-term mean.

We will now show that given the linear pricing rule, our mean-reversion rates needs to be equal for all our processes. We first assume that $\alpha_2^{2,1} \neq \alpha_1 \neq \alpha_2^{2,2}$, this gives us

$$(F_2^i - m_2^i) = a_1^i (F_1 - m_1).$$

From (4.20), this is true if and only if $F_2 = c_1 m_2^1 + c_2 m_2^2$. This means we have to set the long term mean of $F_2(t)$ equal to its current price. As we expect $F_2(t)$ to change almost surely when time passes, this is not a feasible long term strategy. Therefore we need either $\alpha_2^{2,1}$ or $\alpha_2^{2,2}$ to be equal to α_1. We therefore assume that

$$\alpha_2^{2,1} = \alpha_1,$$

by symmetry the result would be the same for

$$\alpha_2^{2,2} = \alpha_1.$$

Since either $a_2^{2,1}$ or $a_2^{2,2}$ is different from 0, we get from (4.18) or (4.19) that:

$$F_2^2 = m_2^2.$$

From the same argument as earlier, we get that this is an infeasible strategy. We therefore conclude from (4.15) that

$$\alpha_2^{2,1} = \alpha_1 = \alpha_2^{2,2}.$$

By adding more Futures products, we will get more terms of the form

$$(F_i - m_i)e^{-\alpha_i t}.$$

And we get either $F_i = m_i$, which is not feasible, or we need α_i equal to the mean reverting speed of the already observed Futures products. This concludes that the speed of mean reverting has to be equal for all Futures products in our framework.

For the variance of $F_2^i(t)$, we can define this in two different manners, either:

$$Var(F_2^i(t)) = \int_0^t e^{2\alpha(s-t)}(\sigma_2^i(s))^2 ds, \tag{4.21}$$

or

$$\begin{aligned} Var(\tilde{F}_2^i(t)) &= (a_1^{2,i})^2 \int_0^t e^{2\alpha(s-t)}(\sigma_1(s))^2 ds \\ &\quad + (a_2^{2,i}c_1)^2 \int_0^t e^{2\alpha(s-t)}(\sigma_2^1(s))^2 ds \\ &\quad + (a_2^{2,i}c_2)^2 \int_0^t e^{2\alpha(s-t)}(\sigma_2^2(s))^2 ds. \end{aligned} \tag{4.22}$$

This means the uncertainty of the Brownian motions $B_2^1(t), B_2^2(t)$ are spread to the two Futures products $F_2^1(t), F_2^2(t)$. If the variance of each individual product is increasing or decreasing as a result of this, depends on the size of the parameters c_i, $a_2^{2,i}$ and $\sigma_2^i(s)$. In general it will decrease, but if we assume large uncertainty of one process compared to the other, the spillover effect might increase the variance of the process with low variance. We also get more uncertainty, as the product $F_1(t)$ will affect the individual products $F_2^1(t)$ and $F_2^2(t)$ because of the spillover effect.

The calculations done until this point are done assuming we are sure about the starting points of F_2^i, but in reality these are just coming from our PFC. When these products are added to the market, we expect the price to be equal to the price coming from the PFC, in reality we will have some uncertainty of this price as well. In the following we will investigate this uncertainty.

This shows that all processes need the same mean-reversion speed α if we model the atomic Futures as OU-processes, and we have a linear pricing rule. And we get the long-term mean for each process, given a long-term mean of the currently observed products. If we had different mean reverting factors, and we would like to estimate what the price of a specific load curve is at some point T in the future, we would need to have different parameters for every atomic Futures product, while when the mean reversion speeds are equal, we can estimate the products that are traded at time T. We also get this expressions for the difference between the processes $F_2^1(t) - \tilde{F}_2^1(t)$:

$$\begin{aligned} F_2^1(t) - \tilde{F}_2^1(t) &= (1 - a_2^{2,1}c_1) \int_0^t e^{\alpha(s-t)}\sigma_2^1(s) dB_2^1(s) \\ &\quad - a_1^{2,1} \int_0^t e^{\alpha(s-t)}\sigma_1(s) dB_1(s) - a_2^{2,1}c_2 \int_0^t e^{\alpha(s-t)}\sigma_2^2(s) dB_2^1(s) \end{aligned} \tag{4.23}$$

so the probability for that the price process $\tilde{F}_2^1(t)$ is wrong with respect to how we price our PFC is normally distributed, with following parameters:

$$\begin{aligned} F_2^1(t) - \tilde{F}_2^1(t) \quad &\sim \quad \mathcal{N}\left(0, \int_0^t e^{2\alpha(s-t)}\left((1-a_2^{2,1}c_1)^2(\sigma_2^1(s))^2\right.\right. \\ &\left.\left. +(a_1^{2,1})^2(\sigma_1(s))^2 + (a_2^{2,1}c_2)^2(\sigma_2^2(s))^2\right)ds\right). \end{aligned} \tag{4.24}$$

The calculations we have done until now for the mean-reverting speed are independent of the fact that we use a Brownian motion to drive the uncertainty. We could therefore use a general Lévy process to model the uncertainty of our OU-process and get the same results for the mean reverting factor α.

4.4 Uncertainty of $F_2^i(0)$

Until now we have discussed how the dynamics of different Futures products compare to each other when working with processes driven by a OU-process, but this is only one part of the uncertainty. We have until now assumed we observe the Futures products F_1, F_2, and from this calculated implied prices $\tilde{F}_2^1, \tilde{F}_2^2$ for the products F_2^1, F_2^2 as

$$\tilde{F}_2^i = s_2^i + a_1^{2,i}F_1 + a_2^{2,i}F_2.$$

As we do not know if the PFC correctly estimates these prices, we will in the following we will assume some uncertainty of F_2^i. If we assume $\tilde{F}_2^i$ is normally distributed, we get the expected price given by the PFC:

$$s_2^i + a_1^{2,i}F_1 + a_2^{2,i}F_2,$$

and the variance:

$$(\sigma_i)^2.$$

By other Lévy processes we can have more moments than only mean and variance. We will assume that the Lévy process driving the process and the starting point follows the same distribution. In that case we get the distribution of our process is equal to the sum of the Lévy process and the distribution of the starting point. Since the Lévy process is per definition infinitely distributed, the sum of these distributions has the same distribution as the Lévy process, but with different parameters, if the uncertainty of the starting point and the Lévy process driving the Futures product are independent. We will in general work with Brownian motions, but in Appendix B we will discuss some other Lévy processes, and why these processes are not viable. We will assume that F_2^i is independent of the Brownian motions driving our processes.

This uncertainty corresponds to the uncertainty of buying a portfolio of Futures products corresponding to the implied Futures product $F_2^i(0)$. We have previously shown that instead of buying the implied product $F_2^i(0)$ from the PFC, we can buy a corresponding portfolio of the observed Futures products. When the number of observed Futures products remain constant, this portfolio will hedge the implied product perfectly. When this implied product is added to the market, there will almost surely be a difference between the price from the PFC and the observed market price, and our uncertainty corresponds to this risk.

4.4.1 General Framework

In the following we will consider only two products F_1 and F_2, and we will assume we observe the weighted average $c_1F_2 + c_2F_2 = F$. We price

$$\tilde{F}_i = s_i + a_i F.$$

We will work as follows: Today we assume that the price for F_i is given by $s_i + a_1^i F_1$, and at any point in time t in the future this is a random variable following the distribution given by the stochastic process driving $F_i(t)$. At time $t < T_1$ we observe the sum $F(t) = c_1F_1(t) + c_2F_2(t)$. By observing this sum, we get more information of the processes $F_2^i(t);\ i = 1, 2$. This in the same sense as if one throws two dice, but only gets told what the sum of these two dice are, this sum tells something of the possible outcomes of the individual die. We can calculate such implied distributions by Bayes formula:

$$f_{F_i}(x|c_1F_1 + c_2F_2 = z) = \frac{f_{c_1F_1+c_2F_2}(z|F_1 = x)f_{F_1}(x)}{f_{c_1F_1+c_2F_2}(z)}. \tag{4.25}$$

Here

$$f_{c_1F_1+c_2F_2}(z) = \int_{-\infty}^{\infty} f_{c_1F_1+c_2F_2}(z|F_1 = x)f_{F_1}(x)dx. \tag{4.26}$$

Since we know that we will price $F_i(t)$ by:

$$F_i(t) = s_i + a_1^i F(t),$$

a natural condition for our probability distribution $f_{F_1}(x|c_1F_1 + c_2F_2 = z)$ is that:

$$E[F_i(t)|c_1F_1(t) + c_2F_2(t) = z] = s_i + a_i z.$$

This says that if a probability distribution should be feasible for our framework, we need an analytic solution of the mean of our implied distribution. We need this mean to be linear in z, which is the weighted sum of our two Futures products. We will in the following show this is the case for normally distributed random variables, in B we show this is not the case for our other proposed Lévy processes.

Theorem 1. *Assume we have two normally distributed variables X_1, X_2, where:*

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

then we denote the weighted sum of these two random variables as Z:

$$Z = c_1X_1 + c_2X_2$$

If one observes that the value of the random variable Z is equal to z, then the distribution of X_i, given this sum, written as $X_i|Z$, is given as follows:

$$X_i|Z \sim \mathcal{N}(\mu_i|Z, \sigma_i^2|Z)$$

where:

$$\mu_i|Z = \frac{zc_i\sigma_i^2 - c_1c_2\mu_j\sigma_i^2 + \mu_i c_j^2\sigma_j^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2} \tag{4.27}$$

and:

$$\sigma_i^2|Z = \frac{2c_j^2\sigma_1^2\sigma_2^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}. \tag{4.28}$$

Where as earlier $j = 2$, when $i = 1$.

Proof. See Appendix B.

□

If we observe that $Z = z$, and we write z as $z = c_1\mu_1 + c_2\mu_2 + k$, which is the expected value of Z plus some number k we get:

$$\mu_i|Z = c_1\mu_1 + c_2\mu_2 + k = \mu_i + \frac{k\sigma_i^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}$$

So the difference between what we expect our random variable Z to be, $c_1\mu_1 + c_2\mu_2$, and what we observe it to be, $c_1\mu_1 + c_2\mu_2 + k$, is redistributed to F_1 and F_2 dependent on the relative variance of these two processes. It should be noted that the uncertainty of $\sigma_i^2|Z$ is independent of z.

4.4.2 Mathematical Model

We will in the following work with two OU-processes $F_1(t)$ and $F_2(t)$, where:

$$dF_i(t) = \alpha(m_i - F_i(t))dt + \sigma_i(t)dB_i(t); F_i(0) = F_i \tag{4.29}$$

and for $0 \leq t < T$ we observe the weighted average of these two processes $c_1F_1(t) + c_2F_2(t) = Z(t)$, and we price F_i by

$$F_i = s_i + a_iZ(t)$$

which means the starting point is $F_i(0) = s_i + a_iZ(0)$, and we assume this is normally distributed $F_i(0) \sim \mathcal{N}(s_i + a_iZ(0), \sigma_i^2)$. We work with the notation Z and z for the sum of our products instead of simply $F(t)$, as we will in the following have several Futures products, and we will clearly distinct $Z(t)$ from these individual products. We use Z when we talk about the random variable, and we use z when we talked about the observed sum.

Our SDEs have solutions of the form

$$\begin{aligned} F_i(t) &= F_ie^{-\alpha t} + (1 - e^{-\alpha t})m_i \\ &\quad + e^{-\alpha t}\int_0^t e^{\alpha_s}\sigma_i(s)dB_i(s), \end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
Z(t) &= c_1F_1(t) + c_2F_2(t) \\
&= c_1F_1(0)e^{-\alpha t} + c_2F_2(0)e^{-\alpha t} \\
&\quad + c_1(1-e^{-\alpha t})m_1 \\
&\quad + c_2(1-e^{-\alpha t})m_2 \\
&\quad + c_1e^{-\alpha t}\int_0^t e^{\alpha s}\sigma_1(s)dB_1(s) \\
&\quad + c_2e^{-\alpha t}\int_0^t e^{\alpha s}\sigma_2(s)dB_2(s) \\
&= Z(0)e^{-\alpha t} + m_z(1-e^{-\alpha t}) \\
&\quad + e^{-\alpha t}\int_0^t e^{\alpha s}(c_1\sigma_1(s)dB_1(s) + c_2\sigma_2(s)dB_2(s)).
\end{aligned}$$

Where $m_z = c_1m_1 + c_2m_2$ is the long-term mean of the process Z. We can define

$$\tilde{B}(t) = \int_0^t \left(\frac{c_2\sigma_2(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}dB_2(s) + \frac{c_1\sigma_1(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}dB_1(s) \right)$$

which is again a Brownian motion, and we can rewrite $Z(t)$ as

$$\begin{aligned}
Z(t) &= Z(0)e^{-\alpha t} + m_z(1-e^{-\alpha t}) \\
&\quad + e^{-\alpha t}\int_0^t e^{\alpha s}\sigma_z(s)d\tilde{B}(s)
\end{aligned}$$

where $\sigma_z(s) = \sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}$. The covariance between our new Brownian motion $\tilde{B}(t)$ and $B_i(t)$ is given by:

$$\begin{aligned}
E[\tilde{B}(t)B_i(t)] &= E\left[\int_0^t \int_0^t \left(\frac{c_2\sigma_2(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}dB_2(s) \right.\right. \\
&\quad \left.\left. + \frac{c_1\sigma_1(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}dB_1(s)\right) \cdot \int_0^t dB_i(s)\right] \\
&= E\left[\int_0^t \frac{c_i\sigma_i(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}dB_i(s) \cdot \int_0^t dB_i(s)\right] \\
&= \int_0^t \frac{c_i\sigma_i(s)}{\sqrt{c_1^2\sigma_1^2(s) + c_2^2\sigma_2^2(s)}}ds
\end{aligned}$$

We will then see what we can say about the individual processes, if one observes the mean of them. If one observes only the quarterly Futures products, one will use the framework described earlier to derive the PFC, which again gives us an estimate for the monthly Futures. This estimate will be a linear combination of the observed quarterly Futures. If one assumes a distribution for each Futures product, then one gets an implied distribution given the sum of these Futures products, as shown earlier. In the following we will work with the quantities:

$$E[F_i(t)|Z(t) = z],$$

and

$$Var(F_i(t)|Z(t) = z).$$

From this we will get more insight in the parameters of $F_1(t)$ and $F_2(t)$. As we have already studied what happens with the parameters determining the expected price and mean reversion, we will focus our attention on our volatility.

In this framework we have several ways to describe the variance, which are all closely linked. We will first list and describe the various variance measures we will use in the following:

$\sigma_i^2(t)$: Is the variance function in our OU-process
$\sigma_{0,i}^2$: Is the variance of the uncertainty of our price F_i coming from the PFC.
$\bar{\sigma}_i^2(t) = Var(F_i(t))$: Is the total variance of our OU process at time t, as seen from time $t = 0$. Is a function of $\sigma_i^2(t)$ and $\sigma_{0,i}^2$
$\bar{\sigma}_i^2(t)|Z$: Is the total variance of our OU-process at time t when we observe the sum $c_1X_1(t) + c_2X_2(t) = Z(t)$. We have that $\bar{\sigma}_i^2(0)|Z = 2c_ja_j\sigma_{0,i}^2$.

Theorem 2. *Given two Ornstein-Uhlenbeck processes $F_1(t), F_2(t)$ with time dependent volatility, and unknown, but normally distributed starting points, which are independent of the Brownian motions, defined as follows:*

$$dF_i(t) = \alpha(m_i - F_i(t))dt + \sigma_i(t)dB_i(t);\ F_i(0) \sim \mathcal{N}(\mu_0^i, \sigma_{0,i}^2)$$

where one at each time t can observe the linear combination of these:

$$Z(t) = c_1F_1(t) + c_2F_2(t)$$

Then the functions $\sigma_1(t), \sigma_2(t)$, need to have the relationship

$$\sigma_1^2(t) = \frac{a_1 \cdot c_2}{a_2 \cdot c_1}\sigma_2^2(t)$$

for the expected value

$$E[F_i(t)|c_1X_1(t) + c_2X_2(t) = z]$$

to be a linear and time independent function of z.

Proof. We first notice from (4.20) that the long-term mean is dependent on the starting point of our processes. Therefore if the PFC miss-prices a currently un-observed Futures product, one might want to re-estimate the long-term mean as well. If we re-estimate the long-term mean, then the difference $F_i - m_i$ will not change when a new Futures product is traded. We also get that $Var(m_i) = Var(F_i) = \sigma_{0,i}^2$. This gives us

$$\bar{\sigma}_i^2(t) = \sigma_{0,i}^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_i^2(s)ds. \tag{4.31}$$

We can also say that we keep the long-term mean constant, even if we wrongly estimate the starting point, this gives us:

$$\bar{\sigma}_i^2(t) = \sigma_{0,i}^2e^{-2\alpha t} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_i^2(s)ds \tag{4.32}$$

We will in the following work with a general $\bar{\sigma}_i^2(t)$ function, and not choose one framework over the other. We will often only do the calculations with one of the frameworks, as the calculations will be similar for both frameworks.

We denote the mean of $X_i(t)$ by $\bar{\mu}_i(t)$, which is the expected value of our process as seen from time $t = 0$. This is defined as follows:

$$\bar{\mu}_i(t) = (\mu_{0,i} - m_i)e^{-\alpha t} + m_i. \tag{4.33}$$

We want our conditional expected value to be linear in z, giving us this equation:

$$E[X_i(t)|c_1X_1(t) + c_2X_2(t) = z] = s_i + a_iz; \ \forall z \in \mathbb{R}, \forall t \geq 0. \tag{4.34}$$

where, as earlier

$$c_1s_1 + c_2s_2 = 0,$$
$$c_1a_1 + c_2a_2 = 1.$$

From (4.27) we get this equation:

$$s_i + a_iz = \frac{zc_i\bar{\sigma}_i^2(t) - c_1c_2\bar{\mu}_j(t)\bar{\sigma}_i^2(t) + c_j^2\bar{\mu}_i(t)\bar{\sigma}_j^2(t)}{c_1^2\bar{\sigma}_1^2(t) + c_2^2\bar{\sigma}_2^2(t)}. \tag{4.35}$$

As earlier, if $i = 1$ then $j = 2$ and vice versa. By splitting up what is dependent on z, and what is not, and reforming the equations, we get:

$$a_i(c_1^2\bar{\sigma}_1^2(t) + c_2^2\bar{\sigma}_2^2(t)) = c_i\bar{\sigma}_i^2(t), \tag{4.36}$$

and

$$s_i(c_1^2\bar{\sigma}_1^2(t) + c_2^2\bar{\sigma}_2^2(t)) = -c_1c_2\bar{\mu}_j(t)\bar{\sigma}_i^2(t) + c_j^2\bar{\mu}_i(t)\bar{\sigma}_j^2(t). \tag{4.37}$$

From (4.36) we get:

$$\bar{\sigma}_1^2(t) = \frac{a_1 \cdot c_2}{a_2 \cdot c_1}\bar{\sigma}_2^2(t)$$

This means $\bar{\sigma}_i^2(t)$ is increasing when a_1 is increasing, which is intuitive as F_i is more variable with respect to Z when a_i is large. $\bar{\sigma}_i^2(t)$ is also decreasing in c_i as a larger c_i means that the delivery length of F_i is long compared to the delivery length of F_j, and a small upward (downward) change in F_i will need a correspondingly large downward (upward) change in F_j.

From (4.37) we get:

$$\begin{aligned} s_i &= a_i\frac{-c_1c_2\bar{\mu}_j(t)\bar{\sigma}_i^2(t) + c_j^2\bar{\mu}_i(t)\bar{\sigma}_j^2(t)}{c_i\bar{\sigma}_i^2(t)} \\ &= c_j(a_j\bar{\mu}_i(t) - a_i\bar{\mu}_j(t)) \end{aligned}$$

by inserting (4.33) we get:

$$s_i = c_j\left(a_j(\mu_{0,i} - m_i)e^{-\alpha t} + a_jm_i - a_i(\mu_{0,j} - m_j)e^{-\alpha t} - a_im_j\right) \tag{4.38}$$

as this should hold for all $t \geq 0$, we get:

$$a_j(\mu_{0,i} - m_i) = a_i(\mu_{0,j} - m_j). \tag{4.39}$$

This is of the same form as in (4.20), just without the impact of some Futures product outside the delivery period of F_1 and F_2 giving a spill over effect. Meaning adding uncertainty on our starting point does not change the previous result for the dependency between the starting point and the long-term mean. From (4.38) and (4.39) we get:

$$s_i = c_j a_j m_i - c_j a_i m_j; \; i = 1, 2.$$

In our framework, s_i is given from the seasonality curve of our PFC, a_i is given by how we adjust our PFC to the Future prices, and also given by how we construct our PFC, while c_i is only indicating the relative length of the different Futures products in comparison to each other. This means we have two equations with two unknowns, and if these equations are linearly independent we can also deduce the long-term mean of our Futures products from our PFC. This is not wanted, as we want to be able to choose our long-term mean independently of our PFC. By multiplying with c_i and c_j we get:

$$c_j s_j = c_j c_i a_i m_j - c_j c_i a_j m_i = -c_i s_i,$$

and since $c_i s_i + c_j s_j = 0$ we get that our equations are linearly dependent, and we can choose our long-term mean independently from our PFC. We do however get that for a given m_i, the value of m_j will be defined as well.

When looking at $\sigma_i^2(t)|Z = V(X_i(t)|c_1 C_1(t) + c_2 X_2(t))$, which is our variance after we observe the weighted average of our products at time t, we get from (4.28):

$$\begin{aligned} \sigma_i^2(t)|Z(t) &= \frac{2c_j^2 \bar{\sigma}_1^2(t)\bar{\sigma}_2^2(t)}{c_1^2\bar{\sigma}_1^2(t) + c_2^2\bar{\sigma}_2^2(t)} \\ &= \frac{2c_j^2 \bar{\sigma}_i^4(t)(a_j c_i)/(a_i c_j)}{c_i^2\bar{\sigma}_i^2(t) + c_j^2\bar{\sigma}_i^2(t)(a_j c_i)/(a_i c_j)} \\ &= 2c_j a_j \bar{\sigma}_i^2(t) \end{aligned} \tag{4.40}$$

more specifically we get:

$$\sigma_i^2(0)|Z(0) = 2c_j a_j \bar{\sigma}_{0,i}^2 \tag{4.41}$$

So the variance after one observes the sum $Z(t) = c_1 X_1(t) + c_2 X_2(t)$ is only a scaled version of the variance as seen from time $t = 0$, which makes sense as the expression (4.28) is not dependent on the value of z. From (4.32) we get:

$$\bar{\sigma}_i^2(t) = \sigma_{0,i}^2 e^{-2\alpha t} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_i^2(s)ds$$

and since

$$\bar{\sigma}_1^2(t) = \frac{a_1 \cdot c_2}{a_2 \cdot c_1}\bar{\sigma}_2^2(t)$$

we get:

$$\sigma_{0,1}^2 = \frac{a_1 \cdot c_2}{a_2 \cdot c_1}\sigma_{0,2}^2 \tag{4.42}$$

and

$$\sigma_1^2(t) = \frac{a_1 \cdot c_2}{a_2 \cdot c_1}\sigma_2^2(t). \tag{4.43}$$

□

From the previous result, we can write $\frac{c_1}{a_1}\sigma_1^2(t) = \frac{c_2}{a_2}\sigma_2^2(t) = \sigma^2(t)$ which means the variance of any Futures product can be written as

$$\sigma_i^2(t) = \frac{a_i}{c_i}\sigma^2(t). \tag{4.44}$$

This means the variance of any product covering a sub-period of a traded object is dependent on the length of this product as well as how this product is priced from the PFC. We get that the variance of a product covering a certain time period is increasing in a_i and decreasing in its relative length c_i. The variance originating from our initial uncertainty $\sigma_{0,i}^2$ is decreasing, while the second term is dependent on $\sigma_i^2(s)$. One could argue that as we do not change how we construct our PFC when time is changing, we should have the same uncertainty for all t, meaning $\sigma_i^2(t)|Z(t)$ should be independent of t. This gives the equation:

$$\sigma_{0,i}^2 = \sigma_{0,i}^2 e^{-2\alpha t} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_i^2(s)ds$$

and by setting $\sigma_i^2(s) = 2\alpha\sigma_{0,i}^2$ we see that:

$$\begin{aligned}\sigma_{0,i}^2 e^{-2\alpha t} + 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_{0,i}^2 ds &= \sigma_{0,i}^2 e^{-2\alpha t} + e^{-2\alpha t}(e^{2\alpha t} - 1)\sigma_{0,i}^2 \\ &= \sigma_{0,i}^2\end{aligned}$$

as wanted. In the framework of (4.31) we get the equation

$$\sigma_{0,i}^2 = \sigma_{0,i} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\sigma_i^2(s)ds,$$

which implies that $\sigma_i(t) = 0\ \forall t$, which is not suitable. We conclude that this condition is not fitting, and we will not impose this sort of condition on our model. By setting $\sigma_i^2(s) = \sigma_0^2\exp(2\kappa_i t)$ as proposed earlier to incorporate the Samuelson effect, we get:

$$\begin{aligned}\bar{\sigma}_i^2(t) &= \sigma_{0,i}^2 e^{-2\alpha t} + e^{-2\alpha t}\int_0^t \sigma_0^2 e^{2(\alpha+\kappa_i)s}ds \\ &= \sigma_{0,i}^2 e^{-2\alpha t} + \frac{\sigma_0^2}{2(\alpha+\kappa_i)}(e^{2\kappa_i t} - e^{-2\alpha t}).\end{aligned}$$

We will not make a defining conclusion on how the function $\bar{\sigma}_i^2(t)$ should be defined. The variance function of any observed Futures product F_i will be $\frac{a_i}{c_i}\sigma^2(t)$. The term a_i is coming from how we would adjust our seasonality curve to the Futures prices assuming we only observe one Futures product, while c_i is the relative length of Futures product i with respect to this observed product.

Consistency of mean reversion level

We have in previous sections discussed how the PFC should be consistent with respect to the observed Futures products. With this we mean if one observes a set of Futures products and construct a PFC from these products, any implied PFC constructed by adding an implied Futures product as given by the original PFC to the construction, should not change the PFC. We have seen this is the case in our Novel curve, to some extent the case in the method proposed in Fleten and Lemming, 2003, while not the case in the method proposed in Benth, Koekkebakker, and Ollmar, 2007.

In the following we will see if this is the case for the long-term mean as well. In the sense, if we observe a new Futures product, will that change the long-term mean. From (4.20) we observe that the difference between the observed/estimated price of a Futures product and its corresponding long-term mean is dependent on the difference between the price of currently observed products and their long-term mean.

If we assume we observe one Futures product, with current price F and long-term mean m_F, and we want to estimate the long-term mean m_i of F_i; $i = 1, 2$, where $c_1F_1 + c_2F_2 = F$ and $c_1m_1 + c_2m_2 = m_F$. Then from (4.39) we get

$$m_i = a_i(F - m_F) + F_i. \tag{4.45}$$

From this one observes that if the PFC correctly estimates F_i, the long-term mean remains constant, while any difference between the estimated price and the observed price will shift the long-term mean correspondingly. We assume here that the price of the sum of the products remains constant, but we miss price them individually. As earlier a_i determines the estimated price F_i, as we have

$$F_i = s_i + a_iF.$$

From this is it clear if the price F changes, the long-term means remain constant so long the price F_i change correspondingly. This shows that when splitting a Futures product in two, the long-term mean should remain unchanged so long our PFC correctly estimates the individual Futures products. We will now show that the same holds for a Futures product with arbitrary delivery period. We now split up F_2 into F_2^1 and F_2^2. When we only observe the price F, the price F_2^i is given by:

$$F_2^i = s_2^i + a^{2,i}F$$

and the long-term mean is given by:

$$F_2^i - m_2^i = a^{2,i}(F - m_F). \tag{4.46}$$

When we observe F_1 and F_2, the price is given by:

$$F_2^i = s_2^i + a_1^{2,i}F_1 + a_2^{2,i}F_2$$

and the long-term mean is:

$$(F_2^i - m_2^i) = a_1^{2,i}(F_1 - m_1) + a_2^{2,i}c_2^i(F_2^i - m_2^i) + a_2^{2,i}c_2^j(F_2^j - m_2^j).$$

In the following we will show that if we correctly price F_1 and F_2, the long-term mean m_2^i also remains constant. From earlier we know that

$$m_i - F_i = a_i(m_F - F),$$

and that

$$c_2^i(F_2^i - m_2^i) + c_2^j(F_2^j - m_2^j) = (F_2 - m_2).$$

This gives us

$$\begin{aligned}(F_2^i - m_2^i) &= (a_1^{2,i}a_1 + a_2^{2,i}a_2)(F - m_F) \\ &= a^{2,i}(F - m_F).\end{aligned} \tag{4.47}$$

As we have from (4.8):

$$(a_1^{2,i}a_1 + a_2^{2,i}a_2) = a^{2,i}.$$

This is the same as in (4.46), which means our long-term mean is consistent with respect to the number of observed Futures products if your PFC is. This means if your PFC correctly estimates unobserved products, then the long-term mean should not change when we observe these products at a later point in time. If we observe that the PFC miss-prices these objects, the long-term mean should be adjusted correspondingly such that the relationship $(m_i - F_i) = a_i(F - m_F)$ still holds. If we overestimate the price of one Futures product, some other product is similarly underpriced and the long-term mean of this product will be correspondingly reduced, leaving us in equilibrium.

4.5 Full Model

In the previous sections we have seen how the parameters of our different OU-processes depend on each other and on the PFC, we will here characterize how a full model look like. Assume we observe n Futures products $F_i;\ 1 \leq i \leq n$. From this we construct a set of n Ornstein-Uhlenbeck processes, $F_i(t);\ 1 \leq i \leq n$, where

$$dF_i(t) = \alpha(m_i - F_i(t))dt + \sqrt{\frac{a_i}{c_i}}\sigma(t)dB_i(t);\ F_i(0) = F_i;\ i = 1, \cdots, n. \tag{4.48}$$

We then construct an OU-process corresponding to a Futures product with an arbitrary delivery period by taking a linear combination of these products

$$dF(t, T_s, T_e) = \sum_{i=1}^{n} a_i[T_s, T_e]\left[\alpha(m_i - F_i(t))dt + \sqrt{\frac{a_i}{c_i}}\sigma(t)dB_i(t)\right].$$

Where the terms $a_i[T_s, T_e]$ shows how much the price of a product covering $[T_s, T_e]$ is affected by product i. These terms come from how we adjust the seasonality curve to the PFC. The starting point is the linear combination of the starting point of the observed Futures products. If the delivery period corresponds with a Futures product $F_k(t)$ that is currently traded, then $a_k = 1$ and $a_i = 0$ for $i \neq k$.

This gives us a stochastic model for a Futures product with arbitrary length when the number of products remain constant. The next question is what happens when these products split up and we observe a finer granularity of products.

4.5.1 Introduction of new Future

Earlier we have discussed how the parameters of our different processes relate to each other, and we have defined how the dynamics of Futures product with arbitrary delivery period is defined. In this section we will investigate what happens after a new product is introduced to the market. At time $t = 0$ we observe the Futures products $(F_1, \cdots, F_n)$. From this we buy an implicit product F_1^1 which we price as earlier:

$$F_1^1 = s_1^1 + \sum_{i=1}^{n} a_i^{1,1} F_i. \tag{4.49}$$

Assume at time $t = T_1$ this product is added to the market. Then the dynamics for this product for $t < T_1$ is

$$dF_1^{1,1}(t) = \sum_{i=1}^{n} a_i^1 dF_i(t) \tag{4.50}$$

where

$$dF_i(t) = \left[\alpha(m_i - F_i(t))dt + \sqrt{\frac{a_i}{c_i}}\sigma(t)dB_i(t) \right]. \tag{4.51}$$

Since we at time $t = T_1$ observe F_1^1, it will follow its own dynamics for $t > T_1$. We introduce the notation $F_1^{1,t,x}(T)$ as the process $F_1^1(T)$, starting in time t at the value x, which is then evaluated at time T. This follows the notation used in Øksendal, 2010 for processes starting at time t in a point x. We will work with $F_1^{1,T_1,F_1^1(T_1)}(t)$ which is defined as follows

$$F_1^{1,T_1,F_1^1(T_1)}(t) = F_1^1(T_1)e^{\alpha(T_1-t)} + (1-e^{\alpha(T_1-t)})m + e^{\alpha(T_1-t)} \int_t^T e^{\alpha(s-T_1)} \sqrt{\frac{a^{1,1}}{c_1^1}}\sigma(s)dB_1^1(s). \tag{4.52}$$

Where $a^{1,1}$ is coming from how we price product F_1^1 from only one observed Futures product, and c_1^1 is the relative weight of this product.

We are then interested in the distribution of $F_1^{1,0,F_1^1(0)}(t)$; $t > T_1$. We know the dynamics before and after T_1 so we will need to glue these parts together. We will do this using the laws for total expectation and total variation, which states that if X and Y are random variables on the same probability space, then

$$E[Y] = E[E[Y|X]], \tag{4.53}$$

and

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X]). \tag{4.54}$$

We then get:

$$E[F_1^{1,0,F_1^1(0)}(t)] = E[E[F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)]] \tag{4.55}$$

where

$$E[F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)] = F_1^{1,0,F_1^1(0)}(T_1)e^{\alpha(T_1-t)} + (1 - e^{\alpha(T_1-t)})m_1^1 \tag{4.56}$$

and

$$\begin{aligned} Var(F_1^{1,0,F_1^1(0)}(t)) &= E[Var(F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)] \\ &\quad + Var(E[F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)]) \end{aligned} \tag{4.57}$$

where

$$Var(F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)) = e^{-2\alpha t} \int_{T_1}^t e^{2\alpha s} \frac{a^{1,1}}{c_1^1} \sigma^2(s)ds.$$

This gives us

$$E[F_1^{1,0,F_1^1(0)}(t)] = E[F_1^{1,0,F_1^1(0)}(T_1)]e^{\alpha(T_1-t)} + (1 - e^{\alpha(T_1-t)})E[m_1^1] \tag{4.58}$$

and

$$\begin{aligned} Var(F_1^{1,0,F_1^1(0)}(t)) &= E[Var(F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1))] \\ &\quad +Var(E[F_1^{1,0,F_1^1(0)}(t)|F_1^{1,0,F_1^1(0)}(T_1)]) \\ &= e^{2\alpha(T_1-t)}\int_{T_1}^{t} e^{2\alpha(s-T_1)}\frac{a^{1,1}}{c_1^1}\sigma^2(s)ds \\ &\quad +Var\left(F_1^{1,0,F_1^1(0)}(T_1)\cdot e^{\alpha(T_1-t)}+(1-e^{\alpha(T_1-t)})m_1^1\right) \end{aligned} \tag{4.59}$$

We use $E[m_1^1]$ instead of m_1^1 as we have seen it is natural to adjust the long-term mean if the PFC miss prices the unobserved Futures product F_1^1. $E[m_1^1]$ is the long-term mean we estimate at time $t=0$, while m_1^1 will be the long-term mean we estimate at time $t=T_1$. For $t<T_1$ we get

$$\begin{aligned} F_1^{1,0,F_1^1(0)}(T_1) &= \sum_{i=1}^{n} a_i^{1,1}\Big[F_i e^{-\alpha T_1}+(1-e^{-\alpha T_1})m_i \\ &\quad +e^{-\alpha T_1}\int_0^{T_1} e^{\alpha s}\sqrt{\frac{a_i}{c_i}}\sigma(s)dB_i(s)\Big] \end{aligned} \tag{4.60}$$

We have

$$\sum_{i=1}^{n} a_i^{1,1}(F_i-m_i)=F_1^{1,1}-m_1^1$$

and we expect our PFC to correctly price the unobserved product $F_11,1(T_1)$, we get

$$E[F_1^{1,0,F_1^1(0)}(T_1)]=F_1^{1,1}e^{-\alpha T_1}+(1-e^{-\alpha T_1})E[m_1^1]$$

giving us

$$\begin{aligned} E[F_1^{1,0,F_1^1(0)}(t)] &= (F_1^{1,1}e^{-\alpha T_1}+(1-e^{-\alpha T_1})E[m_1^1])e^{\alpha(T_1-t)} \\ &\quad +(1-e^{\alpha(T_1-t)})E[m_1^1] \\ &= F_1^{1,1}e^{-\alpha t}+(1-e^{-\alpha t})E[m_1^1]. \end{aligned} \tag{4.61}$$

Following this framework, we let the difference $\sum_{i=1}^n a_i^{1,1}(F_i-m_i)=F_1^{1,1}-m_1^1$ be constant, and we get uncertainty in our long-term mean $\sum_{i=1}^n a_i^{1,1}m_1^1=E[m_1^1]$. We write

$$(*)=Var\left(F_1^{1,0,F_1^1(0)}(T_1)\cdot e^{\alpha(T_1-t)}+(1-e^{\alpha(T_1-t)})m_1^1\right)$$

And we set $Var(m_1^1) = (\sigma_{0,1}^1)^2$ as in (4.31). This gives us

$$\begin{aligned}
(*) &= Var\Bigg(e^{\alpha(T_1-t)}\sum_{i=1}^{n} a_i^{1,1}\Bigg[F_i e^{-\alpha T_1} + (1-e^{-\alpha T_1})m_i \\
&\quad + e^{-\alpha T_1}\int_0^{T_1} e^{\alpha s}\sqrt{\frac{a_i}{c_i}}\sigma(s)dB_i(s)\Bigg] + (1-e^{\alpha(T_1-t)})m_1^1\Bigg) \\
&= Var\Bigg(e^{\alpha(T_1-t)}\sum_{i=1}^{n} a_i^{1,1}\Bigg[m_i + e^{-\alpha T_1}\int_0^{T_1} e^{\alpha s}\sqrt{\frac{a_i}{c_i}}\sigma(s)dB_i(s)\Bigg] \\
&\quad + (1-e^{\alpha(T_1-t)})m_1^1\Bigg) \\
&= (\sigma_{0,1}^1)^2 + \sum_{i=1}^{n}\left((a_i^{1,1})^2 e^{-2\alpha T_1}\int_0^{T_1} e^{2\alpha s}\frac{a_i}{c_i}\sigma^2(s)ds\right). \qquad (4.62)
\end{aligned}$$

Where we have used that $m_1^1 = \sum_{i=1}^n a_i^{1,1} m_i$. We then obtain

$$\begin{aligned}
Var(F_1^{1,0,F_1^1(0)}(t)) &= e^{-2\alpha t}\int_{T_1}^{t} e^{2\alpha s}\frac{a^{1,1}}{c_1^1}\sigma^2(s)ds \\
&\quad + Var\left(F_1^{1,0,F_1^1(0)}(T_1)\cdot e^{\alpha(T_1-t)} + (1-e^{\alpha(T_1-t)})m_1^1\right) \\
&= e^{-2\alpha t}e^{-2\alpha t}\sum_{i=1}^{n}\left((a_i^{1,1})^2\int_0^{T_1} e^{2\alpha s}\frac{a_i}{c_i}\sigma^2(s)ds\right) \\
&\quad + (\sigma_{0,1}^1)^2 + \int_{T_1}^{t} e^{2\alpha s}\frac{a^{1,1}}{c_1^1}\sigma^2(s)ds). \qquad (4.63)
\end{aligned}$$

Expression (4.63) consists of three terms: The first term corresponds to the uncertainty in all observed Futures products from time $t = 0$ to $t = T_1$. The second term corresponds to the uncertainty if the price coming from the PFC is correct. The third term corresponds to the uncertainty from $t = T_1$ to $t = T$.

From this we get that the distribution of our PFC is normally distributed. For such frameworks we get nice formulas for option prices, in Appendix C we show how to price a European call option, when the underlying is an OU-process.

4.5.2 Comparison to Spot Price Model

The model we have constructed here, should not be considered as a model for the spot-prices, even if there are similarities as both gives a distribution of the price of electricity at some point in the future. Benth, Kallsen, and Meyer-Brandis, 2007 propose a model for the spot prices $S(t)$ as the sum of a seasonality function $\mu(t)$ and a stochastic process $X(t)$

$$S(t) = \mu(t) + X(t), \qquad (4.64)$$

where $X(t)$ is a sum of non-Gaussian Ornstein-Uhlenbeck processes. In such a framework each day (or hourly) price in the futures is a random variable, possibly dependent on the price today. In our framework, as we have the linear framework, we get the whole curve given n Futures prices. That means if we have the seasonality curve, and the price of n independent days, we can from these n days compute the individual Futures prices and from these again compute the whole PFC, if one knows the method used to construct the adjustment function.

How to find n independent days is off course dependent on the method used to adjust the curve. For the method by Biegler-König and Pilz, 2015, the prices in a given time period is only dependent on the Futures product covering this period. Therefore there is a one to one correspondence between each Futures price and each day price in that period. For methods with more spillover effect the situation will be different, but one can in the same sense work backwards to find the Futures prices from a set of daily prices.

Therefore one should not use this framework as a framework for modeling spot prices, the framework suggested here is for estimating of the PFC will be in the future. Such a framework is useful for retailers who sell large amounts of their produced electricity by OTC-contracts priced by the PFC. Retailers who mainly trade from the PFC, will be more interested in how the PFC evolves in time, than what the spot prices will be, as they will minimize their exposure to highly volatile spot prices, by selling their expected produced electricity before entering the spot-marked.

4.5.3 Distribution of $F_i(0)$

It is clear that since we can't observe the starting price of all Futures prices, we can only get an estimate for the ones we do not observe, the expectation of this will come from the PFC, but this tells us nothing about the uncertainty. How to quantify this uncertainty is not obvious.

We can obtain one estimate of this uncertainty by calculating the average price of the realized spot prices over different deliver periods and consider these as Futures prices. Then we can construct a PFC to this set of Futures prices. The difference between the realized prices and this PFC are then de-seasonalized and the distribution of these residuals can be seen as the distribution of our uncertainty. Similar tests can also be done to check the monthly Futures prices against the quarterly prices, and quarterly products against yearly products.

In plot (4.1) we observe the QQ plot of the realized spot prices against the PFC as a test on normality of this data. The QQ plot do approximately lie on a straight line in all cases, so it is not unreasonable that the residuals are normally distributed and it is therefore not unreasonable that our uncertainty of our starting point is normally distributed. We denote our daily residuals for R_d, where

$$R_d \sim \mathcal{N}(\mu_r, \sigma_r^2)$$

and the month residual is the mean of n such residuals, meaning the month residual, denoted M_R is defined as:

$$M_R = \frac{1}{n}\sum_{d=1}^{n} R_d$$

and the distribution of M_R is then:

$$M_R \sim \mathcal{N}(\mu_r, \frac{\sigma_r^2}{n}).$$

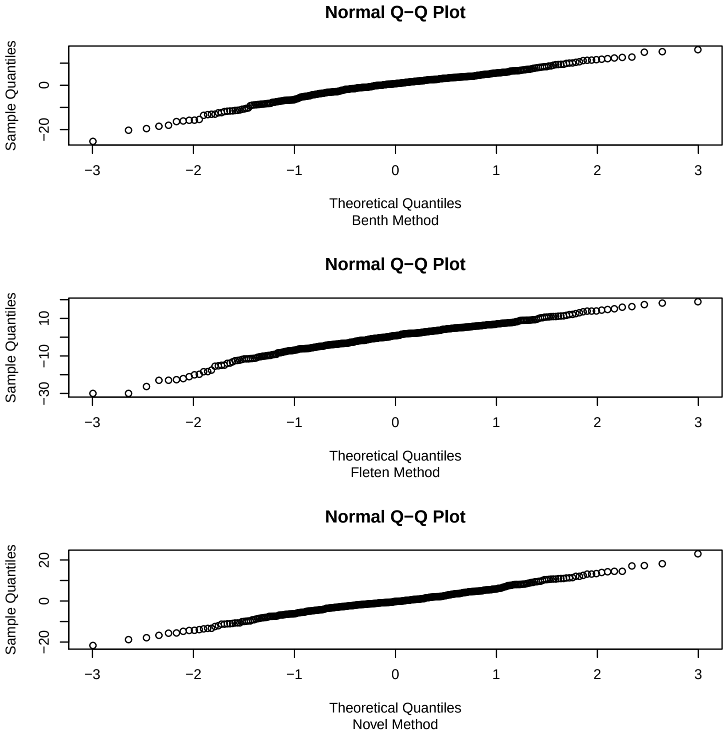

FIGURE 4.1: A QQ-plot of the difference between our realized spot prices and the PFC constructed with 12 monthly Futures products taken as the mean of said spot prices. The QQ plot is with respect to the normal distribution.

Off-course more sophisticated exists for checking whether a data set fits to a certain distribution. We can also do more extensive tests using more data, and different combination of Futures products. It might for example be the data when we observe 12 monthly products is normally distributed, while when only observing 4 quarterly products another distribution fits better. This is just to illustrate that the normal distribution might be appropriate, and that our modeling assumptions are then also appropriate.

4.6 Conclusion

This section treats the subject of how to construct a stochastic model for the PFC as a transformation of a stochastic model for the Futures prices. The basis of the study is given a stochastic model, or a distribution, of the Futures prices at some point in the Futures, we consider the distribution of the HPFC as a linear combination of the distributions of the individual Futures products. Next we conclude that the distribution we consider should be infinitely divisible, as the products we consider are cascading, and we in the Future will possible need to consider more products that are independent than what we observe today. We therefore restrict our study to Ornstein-Uhlenbeck processes driven by a general Lévy process.

We next conclude that a necessary condition of the distribution of our process is that the conditional mean, defined as

$$E[X_1|c_1X_1 + c_2X_2 = z] = a_1 + b_1z,$$

as our PFC is linear in Futures prices. Thereafter we consider several Lévy processes, but as we can't find analytical solutions to the conditional mean, we restrict the further studies to OU-processes driven by Brownian motions. We assume the PFC is build up by a sum of atomic OU-processes, where the delivery length of these are unspecified, but corresponds to the length of the smallest used Futures product. We assume today we only observe a linear combination of these, and therefore the starting point of each atomic OU-process is currently unknown, but estimated from the PFC. We also assume the mean reversion speed α_i and volatility process is $\sigma_i(t)$ is the same before and after we observe the individual atomic products in the market.

From this and from the fact that we want our stochastic PFC should be consistent with respect to our previously shown linear relationship between the observed Futures prices and the PFC we get that the mean reversion speed is constant for all processes, and that the volatility of each single process $F_i(t)$ has this relationship:

$$\sigma_i^2(t) = \frac{a_i}{c_i}\sigma^2(t),$$

where $\sigma(t)$ is some volatility function used as reference. This means that the sum of our atomic OU-processes is again an OU-processes, which makes forecasting easier. For the long-term mean m_i, we get that

$$m_i - F_i = a_i(m_F - F),$$

and correspondingly when we observe more than one Futures product F. As we have this relationship between the current estimated price F_i from the PFC and the long-term mean m_i, we conclude it might be natural to change the long-term mean when we observe the real price of F_i, if our PFC miss-prices this product.

From this we get a characterization of the stochastic processes driving our PFC. In the last step we characterize what the probability distribution of our PFC is at some point T_1 in the future. We consider two cases: One where the number of Futures product observed at T_1 is the same as what we observe today. And one where we assume that at the point $T < T_1$ a Futures product is cascading, and we therefore observe more products at time T_1 than we do today.

This section is based on a framework where the relationship between the Futures products and the PFC is linear. By assuming a non-linear relationship, as is the case in Hagan and West, 2006 this framework will not work. As the Ornstein-Uhlenbeck process used in this framework allows for negative prices, it might be preferable to consider a different framework where other processes might be viable.

Chapter 5

Summary, Conclusion and Further Research

We will here follow with a short conclusion and summary of the results from our thesis, and then talk about possible further research that can be done on the subjects treated in this thesis.

5.1 Summary and Conclusion

In this thesis we focus on the construction of the (hourly) price forward curve for electricity markets. We start by doing a comparison of different construction methods for the PFC, where we focus on two methods from the literature and one novel method. We split our comparison into two parts; first focusing on the seasonality curve and secondly focusing on the adjustment function. The typical seasonality function is either constructed in a functional form, resulting in a smooth curve, or by dummy variables which better model specific characteristics of electricity prices, but lead to large jumps when moving between periods covered by distinct dummy variables. We conclude that a mixture between both a functional and a dummy variable approach is appropriate as we occasionally observe large jumps between consecutive days/hours, but these are exceptions and not the rule. We suggest either starting with an un-smooth curve based on dummy variables, which is later smoothed by the method proposed in Fleten and Lemming, 2003, or to construct a smooth curve based on functions, where exceptions with expected large price jumps, are taken care of in an ad-hoc step by dummy variables. Typical periods where we observe jumps are when new power plants are used in the production to take care of the increased peak-load. Producers are because of this often willing to take temporarily losses for one hour, to have their power plant ready for production when demand is high. We also observe that holiday periods and weekends lead to rapid expected price changes, and in such cases a smooth curve is not sufficient.

For the adjustment function we focus on three methods: 1. The method in Fleten and Lemming, 2003, where we smooth the seasonality curve while fitting the curve to the observed Futures prices. 2. The method considered in Benth, Koekkebakker, and Ollmar, 2007, where they model the difference between the seasonality curve and the observed Futures products by a polynomial spline. 3. The novel method, where we do a constrained least square, fitting the seasonality curve simultaneously to the observed historical prices and the observed Futures prices simultaneously.

We conclude in this section that all three methods have their strengths and weaknesses. The Fleten method has a tendency to suppress the daily/hourly seasonality as we also smooth the seasonality curve. We conclude it is better to re-apply the daily/hourly seasonality after smoothing the seasonality curve, or as we show later, only use this method to adjust the curve without smoothing the seasonality pattern. The Benth method does not smooth the curve, and therefore we keep the daily/hourly pattern. The weakness with this method is that the number of parameters is dependent on the knots used in the construction, leading to an arbitrage opportunity when new products are traded. In the novel method we keep the number of parameters constant, this leads to more free variables when few products are observed, which again leads to a higher chance of over-fitting the seasonality curve when few Futures products are observed.

In the last part of our first section we test these different models to data. We here conclude that the Fleten method, where we re-apply the daily/hourly seasonality, is the best, closely followed by the novel method, for our out-of-sample analysis. For the in-sample analysis we conclude that our novel method is the best, but we also conclude that this is probably because this method has more free variables for the seasonality curve, leading to a better fit. Even though we have a slightly better fit in certain cases by a certain model, some other test might lead to a different result, and we will not conclusively say that one model outperformed the others.

In the second section we do an analysis of how the before-mentioned curves evolve in time. We keep the seasonality curve constant and only update the adjustment function as the Futures used as input change in price and granularity. We show that for all the methods in question the PFC is linear in the Futures products when the number of Futures products remain constant. This simplifies the analysis as the effect of a change in a certain Futures product is not dependent on the price level of the currently observed products. We focus on two parts, what happens when the price of an already observed product changes, and what happens when an observed Futures product cascades into several products with shorter delivery periods.

When the price of a currently observed product changes, we split the effect this change has on the PFC into two parts: How it affects prices covered by this product, and how it affects prices not covered by this product, which we call the spillover-effect. For the method by Fleten and the novel method, the effect a change in the Futures prices has on the curve is relatively independent of the number of products used in the construction, while the increasing number of parameters in the method by Benth makes these dynamics quite dependent on the number of used products. We observe that in the Benth method, when changing the price of certain products, one might get a negative effect on prices in the period covered by this product, which seems unreasonable.

Both the novel method and the method in Benth, Koekkebakker, and Ollmar, 2007 has a relatively large spillover effect, in Table 3.2 at page 61 we see that these methods can have a sensitivity of over 2 with respect to the Futures prices, while the Fleten method has a sensitivity of 1.09. This means that if the price of a certain Futures product increases by 1, the total change in the curve is more than 2 as a result of the spill over effect. We conclude that the best starting point for the adjustment function follows the lines of the Fleten method, which has minimal spillover effect. We also conclude that if all days should be affected by a change in the Futures prices, a certain spillover effect is needed to keep the curve continuous.

In the last section of this thesis we use the before-mentioned linear framework between the PFC and the Futures prices to establish a stochastic model for the PFC given a stochastic model for the Futures. We first conclude that we can only work with processes that follow some infinitely divisible distribution, as we want to give distribution of products with all lengths, and not only quarterly or monthly products. We therefore consider Ornstein-Uhlenbeck processes driven by Lévy processes. With this framework we assume that at each time point t we observe a linear combination of Lévy processes, and we price each of these Lévy processes linearly. We therefore conclude that the distribution we use should be linear in its mean, when conditioned on observing the mean of a sum of such processes. We are only able to calculate this mean analytically for normal distributions, so we focus our work on processes driven by Brownian motions.

We assume the stochastic model for any observed product i is of the form

$$dX_i(t) = \alpha_i(m_i - X_i(t))dt + \sigma_i(t)dB_i(t).$$

We conclude that the mean reversion speed of all individual product must be the same

$$\alpha_i = \alpha; \forall i.$$

For the volatility we conclude that

$$\frac{c_i}{a_i}\sigma_i^2(t) = \sigma^2(t); \forall i.$$

Here c_i is the relative length of product i and a_i tells us how dependent the price of product i is with respect to the price of the whole period. For the long-term mean we get equations of the form

$$(m_2^i - F_2^i) = \sum_{j=1}^{n} a_j^{2,i}(m_j - F_j)$$

This means the difference between the long-term mean and the currently estimated price of an un-observed product should be dependent on the difference between the current price and long-term mean of the currently observed products. As we have this relationship, we conclude it might be reasonable to re-estimate the long-term mean as well, if the PFC does not correctly price a Futures product that is added to the market.

5.2 Further Work

In this thesis we have discussed the construction of the PFC and how this object evolves in this, and from this again constructed a novel framework for a stochastic model for the PFC. For further research it would be interesting to further develop the framework for the adjustment function, and also for the stochastic model for the PFC. Possible extensions would be:

- In our stochastic model our main assumption is that the dynamics of the quarterly product is the same before and after we observe the individual monthly products. For later research, observing how the quarterly prices behave before and after the inclusion of monthly products in the market would be an interesting study, to verify if this assumption holds. By investigating this, we might get input on how to improve our model.

- In this thesis we reject all models apart from the model based on the Brownian motion, because we can't find analytical solutions to the implied expected value. Further research investigating if we can find the analytical solution to the implied expected value for the proposed Lévy processes, or possibly other Lévy processes, will give us more possibilities to develop more advanced models that can fit better to the data.

- In this model we assume a linear relationship between the Futures and the PFC, to study other relationships for the adjustment function, like the one in Caldana, Fusai, and Roncoroni, 2017, would give us more possibilities for the adjustment function. The extension of a non-linear framework for the stochastic model would be interesting for further research.

- We have here assumed that our Futures products are independent, in a more realistic setting we would assume some co-variance structure between the products.

Appendix A

Appendix for Construction of HPFC

Fitting of HPFCs

A.0.1 Fletens Method

The methods used in the Papers Fleten and Lemming, 2003 and Benth, Koekkebakker, and Ollmar, 2007 are both based on a quadratic minimization with a linear constraint, not unlike the constrained least squared used in the novel approach. To solve these problems on need to use the Lagrange multiplier method. We will show how to do this for the Fleten method, as the construction of the matrices are similar.

Recall the Fleten method is as follows:

$$\min_f \sum_{i=1}^{T}(f_i - B_i)^2 + \lambda \sum_{i=2}^{T-1}(f_{i-1} - 2f_i + f_{i+1})^2$$

given

$$\sum_{i \in F_j} f_i = P_j, \; \forall j$$

By using the Lagrange multiplier method, we get this Lagrangian:

$$L(f, \delta) = \sum_{i=1}^{T}(f_i - B_i)^2 + \lambda \sum_{i=2}^{T-1}(f_{i-1} - 2f_i + f_{i+1})^2 + \sum_j \sum_{t \in F_j} \delta_j (f_t - P_j)$$

Where F_j is the days when Futures product j is traded, and P_j is the corresponding price of this Futures product.

By differentiating we obtain these equations:

$$\frac{dL}{df_1} = 2(f_1 - B_1) + 2\lambda(f_1 - 2f_2 + f_3) + \delta_1 = 0$$

$$\frac{dL}{df_2} = 2(f_2 - B_2) + 2\lambda(-2f_1 + 2f_2 - 4f_3 + f_4) + \delta_1 = 0$$

$$\frac{dL}{df_3} = 2(f_3 - B_3) + 2\lambda(f_1 - 4f_2 + 6f_3 - 4f_4 + f_5) + \delta_1 = 0$$

$$\frac{dL}{df_t} = 2(f_t - B_t) + 2\lambda(f_{t-2} - 4f_{t-1} + 6f_t - 4f_{t+1} + f_{t+2}) + \delta_. = 0$$

The two first and last terms are different from the remaining terms, since the counter in the smoothing term starts at 2. For the last two terms we obtain similar expressions:

$$\frac{dL}{df_{T-1}} = 2(f_{T-1} - B_{T_1}) + 2\lambda(-2f_{T-3} + 2f_{T-1} - 4f_{T-1} + f_T) + \delta_N = 0$$

$$\frac{dL}{df_T} = 2(f_T - B_T) + 2\lambda(f_1 - 2f_2 + f_3)2\lambda(-2f_{T-2} + 2f_2 - 4f_{T-1} + f_T) + \delta_N = 0$$

It is worth noting that for the three first equations the index on the $\delta.$ parameter is equal to 1. This is because $\delta.$ is the Lagrange multiplier and is dependent on which Futures product is traded. For monthly Futures products, the index will change for approximately every thirty days, and correspondingly for quarterly and yearly products.

$$\frac{dL}{d\delta.} = \sum_{t \in F.}(f_t - P.) = 0$$

These equations can be organized in matrix form:

$$\begin{bmatrix} 2H & A^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ \delta \end{bmatrix} = \begin{bmatrix} B \\ P \end{bmatrix}$$

Where H is the quadratic matrix:

$$H = \begin{bmatrix} 1+\lambda & -2\lambda & \lambda & 0 & 0 & \cdots \\ -2\lambda & 1+5\lambda & -4\lambda & \lambda & 0 & \cdots \\ \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 \\ 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda \\ \cdots & & & & & \end{bmatrix}.$$

A is an $T \times J$-matrix, where J is the number of Futures products in the market, each element in A is 1 if that Futures product is sold on the corresponding day, otherwise it is 0. B is the vector of forecasted prices from the seasonality curve, and P is the vector containing the Futures prices.

A.0.2 Benths Method

As in the method by Fleten and Lemming, 2003, the method by Benth, Koekkebakker, and Ollmar, 2007 can be solved by the Lagrange multiplier method. We will only show the main steps here, for a more thorough presentation we refer to the original paper, where also notes on how it can be extended to bid-ask spreads is covered.

We can then write the minimization problem as:

$$\min_x \mathbf{x^T H x} \tag{A.1}$$

where

$$H = \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix}$$

and

$$h_j = \begin{bmatrix} \frac{144}{5}\Delta_j^5 & 18\delta_j^4 & 8\Delta_j^3 & 0 & 0 & \\ 18\Delta_j^4 & 12\Delta_j^3 & 6\Delta_j^2 & 0 & 0 & \\ 8\Delta_j^3 & 6\Delta_j^2 & 4\Delta_j^1 & 0 & 0 & \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Delta_j^l = t_{j+1}^l - t_j^l,$$

where t_j/t_{j+1} is start/end point of period j. The numbers in the h_j matrix comes from the integral:

$$\int_{t_j}^{t_{j+1}} (\varepsilon''(t,x))^2 dt = \frac{144}{5} a_j^2 \Delta_j^5 + 12 b_j^2 \Delta_j^3 + 4 c_j^2 \Delta_j^1 + 2 \cdot 18 a_j b_j \Delta_j^4 + 2 \cdot 8 a_j c_j \Delta_j^3 + 2 \cdot 12 b_j c_j \Delta_j^2$$

.

Here a_j^2, b_j^2, c_j^2 corresponds to the diagonal elements and the cross-products corresponds to the elements not on the diagonal. H is then a $5n \times 5n$ matrix.

Then the constrains are represented by the matrix equation $Ax = b$, where A is a $(4n-2) \times 5n$ matrix and b is a vector with length $(4n-2)$. $4n-2$ is the number of constraints, we have n constraints for the n Futures prices observed and $3n-2$ constraints for the smoothness of the spline curve. The problem is then solved by this Lagrange multiplier problem:

$$\min_{x,\lambda} \mathbf{x^T H x} + \lambda^{\mathbf{T}}(\mathbf{Ax} - \mathbf{b}) \tag{A.2}$$

with solution $[x^*, \lambda^*]$ given by:

$$\begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 2H & A^T \\ A & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ b \end{bmatrix}$$

When working in high dimensions, one might need to work with numerical methods to invert the matrix on the left-hand side, like QR-factorization.

A.0.3 Novel modeling approach

The correction term $S(\cdot)$ that ensures the continuity is given by the vector defined as:

$$S = (0, -3, 3, 0, 4, -1, 5, 5, -3, 6, -4, 7)$$

where $S(m)$ is element number m in the vector S.

The matrices A needed for the optimization in the novel modeling approach are defined as follows:

$$A = [A_1, \cdots, A_6, B_1, \cdots, B_6, A_4^1, \cdots, A_4^4, B_4^1, \cdots, B_4^4], \tag{A.3}$$

where:

$$A_j = (\sin(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}), \cdots, \sin(\frac{2\pi \cdot j \cdot 31}{12 \cdot 31}), \sin(\frac{2\pi \cdot j \cdot 25}{12 \cdot 28}), \cdots\cdots, \sin(\frac{2\pi \cdot j \cdot 341}{12 \cdot 31})),$$

$$B_j = (\cos(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}), \cdots, \cos(\frac{2\pi \cdot j \cdot 31}{12 \cdot 31}), \cos(\frac{2\pi \cdot j \cdot 25}{12 \cdot 28}), \cdots\cdots, \cos(\frac{2\pi \cdot j \cdot 341}{12 \cdot 31})),$$

and A_j^Q for $1 \leq Q \leq 4$ is the vector A_j in quarter Q, and 0 otherwise, giving:

$$A_j^1 = (\sin(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}), \cdots, \sin(\frac{2\pi \cdot j \cdot 93}{12 \cdot 31}), 0, \cdots\cdots, 0).$$

B_j^Q is equally defined.

The matrix for the constraints, named C is separated into two different matrices, H, G, where H makes sure the PFC fits to the Futures, and G takes care of the continuity of the spline part of the PFC. The constraints coming from the Futures are given as:

$$V_i = \frac{1}{T_E - T_S} \int_{T_S}^{T_E} f(t) dt \tag{A.4}$$

where we divide by the length of the period the Futures product is covering since the Futures price is denoted by the average price for that period. Assuming we have a Futures product covering January with price V_1, the corresponding constraint for term j of the Fourier series is:

$$\begin{aligned} V_1 &= \frac{1}{31} a_j \int_0^{31} \sin(\frac{2\pi \cdot j \cdot t}{12 \cdot 31}) + b_j \cos(\frac{2\pi \cdot j \cdot t}{12 \cdot 31}) dt \\ &= \frac{31 \cdot 12}{31 \cdot 2\pi \cdot j} [-a_j(\cos(\frac{2\pi \cdot j}{12}) - \cos(0)) + b_j(\sin(\frac{2\pi \cdot j}{12}) - \sin(0))] \end{aligned}$$

Which shows two of the advantages with changing the seasonality corresponding to the length of the months: Firstly, one can cancel the terms coming from the dividing by the length of the period directly against the term coming from multiplying with the denominator in the $\sin / \cos$ term. Secondly, one only need to evaluate $\sin / \cos$ in values that are a multiple of $2\pi/12$. By denoting:

$$F(i,j)_C = \frac{12}{2\pi \cdot j} \cos(\frac{2\pi \cdot j \cdot (i+1)}{12}) - \cos(\frac{2\pi \cdot j \cdot (i)}{12})) \tag{A.5}$$

and similar for the $\sin$ function by $F(i,j)_S$. Then the matrix H is defined as follows

$$H = \begin{bmatrix} F(1,1)_C & F(1,2)_C & \cdots & F(1,6)_C & F(1,1)_S & \cdots \\ F(2,1)_C & \cdots & \cdots & \cdots & F(2,1)_S & \cdots \\ \cdots & \cdots & & & & \\ F(12,1)_C & \cdots & & & & \end{bmatrix},$$

If we assume we observe all monthly Futures products. Where the pattern is similar as for the matrix A with the spline coefficients. For the matrix G that ensures continuity and continuity of the derivatives of the splines, one has these constraints:

Continuity:

$$\begin{aligned} & b_4^1 + b_{12}^1 \\ = \; & b_4^2 + b_{12}^2 \\ & b_4^2 + b_{12}^2 \\ = \; & b_4^3 + b_{12}^3 \\ & b_4^3 + b_{12}^3 \\ = \; & b_4^4 + b_{12}^4 \end{aligned}$$

differentiability:

$$
\begin{aligned}
& \frac{1}{31}(a_4^1 + 3a_{12}^1) \\
= \quad & \frac{1}{30}(a_4^2 + 3a_{12}^2) \\
& \frac{1}{30}(a_4^2 + 3a_{12}^2) \\
= \quad & \frac{1}{31}(a_4^3 + 3a_{12}^3) \\
& \frac{1}{30}(a_4^3 + 3a_{12}^3) \\
= \quad & \frac{1}{31}(a_4^4 + 3a_{12}^4)
\end{aligned}
$$

Giving us 6 constraints for the 26 parameters, making G a 6×26 matrix.

Appendix B

Appendix for Implied Distributions

Implied Distributions

B.0.1 Normal Distribution

We will show for X_1, as the proof is identical for the two variables. From Bayes theorem we get that the probability density function for $X_1|Z$ is given by:

$$f_{X_1}(x|c_1X_1 + c_2X_2 = z) = \frac{f_{c_1X_1+c_2X_2}(z|X_1 = x)f_{X_1}(x)}{f_{c_1X_1+c_2X_2}(z)} \tag{B.1}$$

where:

$$f_{c_1X_1+c_2X_2}(z) = \int_{-\infty}^{\infty} f_{c_1X_1+c_2X_2}(z|X_1 = x)f_{X_1}(x)dx \tag{B.2}$$

as the denominator is only the integral of the nominator with respect to x, we will only be interested in the terms that are dependent on x, as everything else will be a normalizing factor.

The function f is the probability density of the normal distribution function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This gives us:

$$\begin{aligned} f_{c_1X_1+c_2X_2}(z|X_1 = x) &= f_{c_1x+c_2X_2}(z) \\ &= \frac{1}{\sqrt{2\pi}c_2\sigma_2}e^{-\frac{(z-(c_1x+c_2\mu_2))^2}{2c_2^2\sigma_2^2}} \end{aligned}$$

since:

$$c_1x + c_2X_2 \sim \mathcal{N}(c_1x + c_2\mu_2, c_2^2\sigma_2^2)$$

and the nominator will be given by:

$$f_{c_1X_1+c_2X_2}(z|X_1 = x)f_{X_1}(x) = \frac{1}{2\pi c_2\sigma_1\sigma_2}e^{-\frac{(z-(c_1x+c_2\mu_2))^2}{2c_2^2\sigma_2^2}-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

As we are only interested in what is dependent of x, we will focus on the exponent:

$$-\frac{(z - (c_1x + c_2\mu_2))^2}{2c_2^2\sigma_2^2} - \frac{(x - \mu_1)^2}{2\sigma_1^2} \tag{B.3}$$

We want to write this on the form:

$$-a(x+b)^2 - c$$

to show that our processes are again normally distributed.

By solving the brackets in expression (B.3) we get:

$$-\frac{z^2 - 2z(c_1x + c_2\mu_2) + c_1^2x^2 + 2c_1c_2\mu_2x + c_2^2\mu_2^2}{2c_2^2\sigma_2^2} - \frac{x^2 - 2x\mu_1 + \mu_1^2}{2\sigma_1^2}$$

which by rearranging of the terms gives us:

$$-(\frac{c_1^2}{2c_2^2\sigma_2^2} + \frac{1}{2\sigma_1^2})x^2 - (\frac{2zc_1 - 2c_1c_2\mu_2}{2c_2^2\sigma_2^2} - \frac{2\mu_1}{2\sigma_1^2})x - \frac{z^2 - 2zc_2\mu_2}{2c_2^2\sigma_2^2} - \frac{\mu_1^2}{2\sigma_1^2}$$

and we want to complete the square, so it can be written in the form:

$$-a(x+b)^2 - c$$

to match it with the normal distribution.

First we calculate the square to see that:

$$-a(x+b)^2 - c \quad = \quad -ax^2 - 2abx - ab^2 - c$$

By matching the terms in front of x and x^2 as well as the constant term, we get:

$$a = (\frac{c_1^2}{2c_2^2\sigma_2^2} + \frac{1}{2\sigma_1^2})$$

$$b = \frac{(\frac{2zc_1 - 2c_1c_2\mu_2}{2c_2^2\sigma_2^2} + \frac{2\mu_1}{2\sigma_1^2})}{(\frac{c_1^2}{c_2^2\sigma_2^2} + \frac{1}{\sigma_1^2})}$$

$$c = \frac{z^2 - 2zc_2\mu_2}{2c_2\sigma_2^2} + \frac{\mu_1^2}{2\sigma_1^2} + ab^2$$

Then as we are only interested in the terms with x or x^2, we ignore the constant term.

From this we get a normally distributed variable $X_1|Z \sim \mathcal{N}(b, 1/a)$. Where:

$$\begin{aligned} b &= \frac{(\frac{2zc_1 - 2c_1c_2\mu_2}{2c_2^2\sigma_2^2} + \frac{2\mu_1}{2\sigma_1^2})}{(\frac{c_1^2}{c_2^2\sigma_2^2} + \frac{1}{\sigma_1^2})} \\ &= \frac{(\frac{zc_1\sigma_1^2 - c_1c_2\mu_2\sigma_1^2 + \mu_1c_2^2\sigma_2^2}{c_2^2\sigma_2^2\sigma_1^2})}{\frac{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}{c_2^2\sigma_2^2\sigma_1^2}} \\ &= \frac{zc_1\sigma_1^2 - c_1c_2\mu_2\sigma_1^2 + \mu_1c_2^2\sigma_2^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2} \end{aligned}$$

and:

$$a = \frac{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}{2\sigma_1^2c_2^2\sigma_2^2}$$

Which implies:

$$E[X_1|c_1X_1 + c_2X_2 = z] = \frac{zc_1\sigma_1^2 - c_1c_2\mu_2\sigma_1^2 + \mu_1c_2^2\sigma_2^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}$$

and:

$$V(X_1|Z) = \frac{2c_2^2\sigma_1^2\sigma_2^2}{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}$$

B.0.2 Other Lévy Processes

We have previously discussed what happens if we look at the conditional probability distribution of a normally distributed random variable, if we know the sum of this and some other independent normally distributed random variable. In the following we will see if similar calculations can be done for other Lévy processes. We will first define what a Lévy process is, and then consider what are natural drawbacks with these processes. Our reasoning for wanting to work with Lévy processes and not only Brownian motions is that we want more flexibility in our modeling frameworks by including skewness, kurtosis and non-continuous sample paths.

A process $F(t)$ is a Lévy process if and only if:

- $F(t)$ has independent increments: for $0 \le t_1 < t_t < t_3 < \infty$, then $F(t_3) - F_i(t_2)$ and $F(t_2) - F(t_1)$ are independent
- $F(t)$ has stationary increments: for $s < t$, $F(t) - F(s)$ has the same distribution as $F(t-s)$
- $F(t)$ is continuous in probability: For any $\epsilon > 0$ and $t \ge 0$, $\lim_{h \to 0} P(|F(t+h) - F(t)| > \epsilon) = 0$

The last condition does not mean that the process $F(t)$ is continuous, just the probability of a jump at a fixed point is very low. The only continuous Lévy process is the Brownian motion. Lévy processes are as the Brownian motion infinitely divisible, since we have independent and stationary increments. We also have that any infinitely divisible distribution corresponds naturally to a Lévy process, for more information on Lévy processes and infinitely divisible distributions see (Sato, 1999). An immediate consequence is that either $F(t) \in \mathbb{R}$ which allows for positive probability for negative prices, or $F(t) \in [0, \infty)$, which implies that $F(t)$ is a subordinator (always increasing). Both of these cases have natural downsides.

We will now see if we can find an analytical solution for

$$f_{F_i}(x | c_1 F_1 + c_1 F_2 = z)$$

where F_i is some infinitely divisible distribution. As there is no general probability density function for all infinitely divisible distributions we will base our study on distributions already used in the literature. We will focus on the hyperbolic distributions which was introduced in the mathematical finance literature in 1997 by Ole E. Barndorff-Nielsen (Barndorff-Nielsen, 1997), for a comprehensive study, see (Barndorff-Nielsen, Mikosch, and Resnick, 2012). We will here focus on the Normal-inverse Gaussian distribution, Variance Gamma, Generalized hyperbolic distribution and the Gamma distribution to include an example of a subordinator.

Normal-inverse Gaussian distribution

The Normal-inverse Gaussian distribution with parameters $\mu, \alpha, \beta, \delta$ and $\gamma = \sqrt{\alpha^2 - \beta^2}$ has PDF:

$$f_{NIG}(x, \mu, \alpha, \beta, \delta) = \frac{\alpha \delta K_1(\alpha \sqrt{\delta^2 + (x-\mu)^2})}{\pi \sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\gamma + \beta(x-\mu)} \tag{B.4}$$

where K_j denotes a modified Bessel function of the second kind:

$$K_\alpha(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\alpha t)\,dt$$

in particular:

$$K_1(x) = \int_{-\infty}^{\infty} \exp(-x(t+\frac{1}{t}))\,dt$$

if c_1X_1 and c_2X_2 are independent random variables that are NIG-distributed with the same values of the parameters α and β , but possibly different values of the location and scale parameters, μ_1, δ_1 and μ_2, δ_2, respectively, then $c_1X_1 + c_2X_2$ is NIG-distributed with parameters $\alpha, \beta, \mu_1 + \mu_2$ and $\delta_1 + \delta_2$.

We are interested in:

$$f_{c_1X_1}(x|c_1X_1 + c_2X_2 = z) = \frac{f_{c_1X_1+c_2X_2}(z|c_1X_1 = x)f_{c_1X_1}(x)}{f_{c_1X_1+c_2X_2}(z)} \tag{B.5}$$

we compute:

$$f_{x+c_2X_2}(z|c_1X_1 = x) = \frac{\alpha\delta_2 K_1(\alpha\sqrt{\delta_2^2 + (z-\mu_2-x)^2})}{\pi\sqrt{\delta_2^2 + (z-\mu_2-x)^2}} e^{\delta_2\gamma+\beta(z-\mu_2-x)} \tag{B.6}$$

$$f_{c_1X_1}(x) = \frac{\alpha\delta_1 K_1(\alpha\sqrt{\delta_1^2 + (x-\mu_1)^2})}{\pi\sqrt{\delta_1^2 + (x-\mu_1)^2}} e^{\delta_1\gamma+\beta(x-\mu_1)} \tag{B.7}$$

$$f_{c_1X_1+c_2X_2}(z) = \frac{\alpha(\delta_1+\delta_2)K_1(\alpha\sqrt{(\delta_1+\delta_2)^2 + (z-\mu_1-\mu_2)^2})}{\pi\sqrt{(\delta_1+\delta_2)^2 + (z-\mu_1-\mu_2)^2}} e^{(\delta_1+\delta_2)\gamma+\beta(z-\mu_1-\mu_2)} \tag{B.8}$$

which gives us:

$$f_{c_1X_1}(x|c_1X_1 + c_2X_2 = z) = \frac{C \cdot K_1(\alpha\sqrt{\delta_2^2 + (z-\mu_2-x)^2}) \cdot K_1(\alpha\sqrt{\delta_1^2 + (x-\mu_1)^2})}{\sqrt{\delta_2^2 + (z-\mu_2-x)^2} \cdot \sqrt{\delta_1^2 + (x-\mu_1)^2}} \tag{B.9}$$

Where:

$$C = \frac{\alpha \cdot \delta_1 \cdot \delta_2 \cdot \sqrt{(\delta_1+\delta_2)^2 + (z-\mu_1-\mu_2)^2}}{\pi(\delta_1+\delta_2)} \tag{B.10}$$

Variance Gamma

The Variance Gamma ($VG(\mu, \alpha, \beta, \lambda, \gamma)$) has PDF:

$$f_{VG}(x, \mu, \alpha, \beta, , \lambda, \gamma) = \frac{\gamma^{2\lambda}|x-\mu|^{\lambda-1/2} K_{\lambda-1/2}\left(\alpha|x-\mu|\right)}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-1/2}}\ e^{\beta(x-\mu)} \tag{B.11}$$

so if $c_iX_i \sim VG(\mu_i, \alpha, \beta, \lambda_i, \gamma)$, then:

$$\begin{aligned} f_{c_1X_1}(x|c_1X_1 + c_2X_2 = z) &= C \cdot |z-\mu_2-x|^{\lambda_2-\frac{1}{2}} K_{\lambda_2-\frac{1}{2}}(\alpha|z-\mu_2-x|) \cdot \\ &\quad |x-\mu_1|^{\lambda_1-\frac{1}{2}} K_{\lambda_1-\frac{1}{2}}(\alpha|x-\mu_1|) \end{aligned} \tag{B.12}$$

where:

$$C = \frac{\Gamma(\lambda_1 + \lambda_2)(2\alpha)^{\frac{1}{2}}}{|z - \mu_2 - \mu_2|^{\lambda_2 - \frac{1}{2}} K_{\lambda_1 + \lambda_2 - \frac{1}{2}}(\alpha|z - \mu_2 - \mu_2|)\Gamma(\lambda_1)\Gamma(\lambda_2)\sqrt{\pi}} \tag{B.13}$$

Generalized hyperbolic distribution

The Generalized Hyperbolic Distribution ($GHD(\lambda, \alpha, \beta, \delta, \mu)$) has PDF:

$$f_{GHD}(x; \lambda, \alpha, \beta, \delta, \mu) = \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi} K_\lambda(\delta\gamma)} e^{\beta(x-\mu)} \frac{K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\left(\sqrt{\delta^2 + (x-\mu)^2}/\alpha\right)^{1/2-\lambda}} \tag{B.14}$$

where $\gamma^2 = \alpha^2 - \beta^2$, if $c_i X_i \sim GHD(\lambda_i, \alpha, \beta, \delta_i, \mu_i)$, then:

$$f_{c_1 X_1}(x|c_1 X_1 + c_2 X_2 = z) = \frac{C \cdot K_{\lambda_2 - 1/2}(\alpha\sqrt{\delta_2^2 + (z - \mu_2 - x)^2}) K_{\lambda_1 - 1/2}(\alpha\sqrt{\delta_1^2 + (x - \mu_1)^2})}{(\sqrt{\delta_2^2 + (z - \mu_2 - x)^2}/\alpha)^{1/2-\lambda_2}(\sqrt{\delta^2 + (x - \mu_1)^2}/\alpha)^{1/2-\lambda_1}} \tag{B.15}$$

where:

$$C = \frac{K_{\lambda_1 + \lambda_2}(\delta\gamma)\sqrt{\delta^2 + z - \mu_1 - \mu_2)^2}/\alpha)^{1/2-\lambda_1-\lambda_2}}{\sqrt{2\pi} K_{\lambda_1}(\delta\gamma) K_{\lambda_2}(\delta\gamma)} \tag{B.16}$$

It should be noted that both the Variance gamma, the NIG and other distributions are special cases of the Generalized Hyperbolic distribution. In particular $GHD(-1/2, \alpha, \beta, \delta, \mu)$ is NIG-distributed and $GHD(\lambda, \alpha, \beta, 0, \mu)$ is Variance Gamma distributed.

There is no work done on implied distributions of this form in the literature, on either the distributions discussed here, or distributions for other Lévy processes. As we do not manage to find analytical solutions to the expected value of these distributions, we will not consider these distributions in the following, and focus on the Brownian motion.

Gamma distribution

As an alternative to the Hyperbolic distributions, we will also look at the Gamma distribution. The Gamma distribution has a support from $[0, \infty)$, which means the corresponding Lévy process is a subordinator. The PDF of the Gamma distribution is given by:

$$f_X(k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta} \tag{B.17}$$

and if

$$c_i X_i \sim Gamma(k_i, \theta)$$

then

$$c_1 X_1 + c_2 X_2 = Gamma(k_1 + k_2, \theta)$$

and we get:

$$f_{c_1 X_1}(x|c_1 X_1 + c_2 X_2 = z) = \frac{\Gamma(k_1 + k_2)}{\Gamma(k_1)\Gamma(k_2)} x^{k_1-1}(z - x)^{k_2-1} z^{1-k_1-k_2} \tag{B.18}$$

So the pdf of $X_1|Z$ is:

$$f_{X_1|Z=z}(k_1,k_2)) = \frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}x^{k_1-1}(z-x)^{k_2-1}z^{1-k_1-k_2};\ 0 \leq x \leq z \tag{B.19}$$

We can compute the expected value of this as:

$$E[X_1|Z] = \int_0^z x\frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}x^{k_1-1}(z-x)^{k_2-1}z^{1-k_1-k_2}dx$$

we use the notation $k_1^* = k_1 + 1$ and that $\Gamma(z+1) = z\Gamma(z)$, and that this is a pdf and integrates to 1.

$$\begin{aligned}
E[X_1|Z] &= \int_0^z x\frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}x^{k_1-1}(z-x)^{k_2-1}z^{1-k_1-k_2}dx \\
&= z\int_0^z \frac{\Gamma(k_1^*+k_2-1)}{\Gamma(k_1^*-1)\Gamma(k_2)}x^{k_1^*-1}(z-x)^{k_2-1}z^{1-k_1^*-k_2}dx \\
&= z\int_0^z \frac{k_1^*-1}{k_1^*+k_2-1}\frac{\Gamma(k_1^*+k_2)}{\Gamma(k_1^*)\Gamma(k_2)}x^{k_1^*-1}(z-x)^{k_2-1}z^{1-k_1^*-k_2}dx \\
&= z\frac{k_1}{k_1+k_2}
\end{aligned}$$

same for the variance:

$$E[X_1^2|Z] = \int_0^z x^2\frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}x^{k_1-1}(z-x)^{k_2-1}z^{1-k_1-k_2}dx \tag{B.20}$$

$$= z\frac{k_1}{k_1+k_2}\int_0^z x^2\frac{\Gamma(k_1^*+k_2)}{\Gamma(k_1^*)\Gamma(k_2)}x^{k_1^*-1}(z-x)^{k_2-1}z^{1-k_1^*-k_2}dx \tag{B.21}$$

$$= z^2\frac{k_1\cdot(k_1+1)}{(k_1+k_2)\cdot(k_1+k_2+1)} \tag{B.22}$$

$$\begin{aligned}
E[X_1^2|Z]-(E[X_1|Z])^2 &= z^2\frac{k_1\cdot(k_1+1)}{(k_1+k_2)\cdot(k_1+k_2+1)} - (z\frac{k_1}{k_1+k_2})^2 \\
&= z^2\frac{k_1k_2}{(k_1+k_2^2)(k_1+k_2+1)}
\end{aligned}$$

So the implied distribution of the Gamma distribution is linear in z, but as this is a subordinator this will still not be an suitable distribution.

Appendix C

Appendix for European Call Option

Option Pricing

In this framework, the asset is a linear combination of normally distributed random variables, which is again a normally distributed random variable. We will denote it by $X \sim \mathcal{N}(\mu, \sigma)$, and our goal is then to calculate option prices on such assets, namely expectations of the form:

$$E[f(X)] \tag{C.1}$$

The standard European call option is given by $f(x) = (x - c)^+$. Then:

$$\begin{aligned} E[(X-c)^+] &= \int_{-\infty}^{\infty} (x-c)^+ P_X(x) dx \\ &= \int_{c}^{\infty} (x-c) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{c}^{\infty} \frac{x-\mu}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx - \int_{c}^{\infty} \frac{c-\mu}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= I_1 - I_2 \end{aligned} \tag{C.2}$$

the first integral is solved by substituting $y = (x - \mu)$, giving us

$$\begin{aligned} I_1 &= \int_{(c-\mu)}^{\infty} \frac{y}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \left[\frac{-2\sigma^2}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \right]_{c-\mu}^{\infty} \\ &= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} e^{-\frac{(c-\mu)^2}{2\sigma^2}}. \end{aligned} \tag{C.3}$$

I_2 is given by the cdf of the normal distribution Φ, which is defined as

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, \tag{C.4}$$

which by symmetry gives:

$$\Phi(-x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt. \tag{C.5}$$

Therefore we have:

$$\begin{aligned}\int_c^\infty \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx &\overset{y=\frac{x-\mu}{\sigma}}{=} \int_{(c-\mu)/\sigma}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{-y^2}{2}}dy \\ &= \Phi(\frac{-c+\mu}{\sigma}), \end{aligned} \tag{C.6}$$

giving us

$$I_2 = (c-\mu)\Phi(\frac{-c+\mu}{\sigma}). \tag{C.7}$$

As a result

$$\begin{aligned} E[(X-c)^+] &= I_1 - I_2 \\ &= \frac{\sqrt{2}\sigma}{\sqrt{\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} + (\mu-c)\Phi(\frac{\mu-c}{\sigma}). \end{aligned} \tag{C.8}$$

We denote $P(\mu,\sigma,c) = E[(X-c)^+]$, and we then want to find out how this is dependent on its parameters. By differentiating we get

$$\begin{aligned} \frac{dP}{d\mu} &= \frac{-(\mu-c)(\sqrt{2}\sigma)}{\sigma^2\sqrt{\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} \\ &\quad + \frac{\mu-c}{\sigma\sqrt{2\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} \\ &\quad + \Phi(\frac{\mu-c}{\sigma}) \\ &= \frac{(c-\mu)}{\sigma\sqrt{2\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} + \Phi(\frac{\mu-c}{\sigma}), \end{aligned} \tag{C.9}$$

$$\begin{aligned} \frac{dP}{dc} &= -\frac{dP}{d\mu} \\ &= \frac{(\mu-c)}{\sigma\sqrt{2\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} - \Phi(\frac{\mu-c}{\sigma}) \\ &= -\frac{dP}{d\mu} \end{aligned} \tag{C.10}$$

and

$$\begin{aligned} \frac{dP}{d\sigma} &= \frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} \\ &\quad + \frac{\sqrt{2}\sigma}{\sqrt{\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}}\frac{(c-\mu)^2}{\sigma^3} \\ &\quad + (\mu-c)\frac{1}{\sqrt{2\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}}\frac{(c-\mu)}{\sigma^2} \\ &= \frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(c-\mu)^2}{2\sigma^2}}\frac{(c-\mu)^2}{\sigma^2}. \end{aligned} \tag{C.11}$$

As $dP/d\mu = -dP/dc$, we will only need to look at one of them. $dP/d\mu$ consists of two parts, one that is always positive, and one that can be both negative and positive. We will first show this will always in sum be positive. By setting $x = \frac{c-\mu}{\sigma}$, we get:

$$\frac{dP}{d\mu} = \frac{x}{\sqrt{2\pi}}e^{-x^2/2} + \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy,$$

and to show that this is positive is equivalent to

$$f(x) = \int_{-\infty}^{-x} e^{-y^2/2}dy + xe^{-x^2/2} > 0.$$

If $x > 0$ this is clear, so it remains to show it for $x < 0$. First we use that:

$$f(0) = \sqrt{\pi/2} > 0$$

afterwards we observe that:

$$\begin{aligned} f'(x) &= -e^{-x^2/2} + e^{-x^2/2} - x^2e^{-x^2/2} \\ &= -x^2e^{-x^2/2} < 0 \end{aligned}$$

this means that our function is decreasing, and we get

$$f(x) > f(0) > 0; \; \forall \; x < 0.$$

Consequently $f(x) > 0; \; \forall x \in \mathbb{R}$.

As for $\frac{dP}{d\sigma}$, this is clearly positive for all $\sigma > 0$, so more uncertainty about the price gives higher price on the call option.

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